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A SIMPLE RULE FOR THE CONSOLIDATION OF ALLOWANCE LISTS*

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The problem of consolidating certain sections of the allowance list is stated and discussed. A practical consolidation rule is proposed and illustrated by numerical examples. The validity of this consolidation rule is established for normally distributed demands, and its general usefulness is outlined.

1. INTRODUCTION

An important set of problems in naval logistics consists of allowance list problems. In this paper we shall be concerned primarily with the consolidation problem, but in the course of devising a solution for it we shall have to deal with a more general allowance list problem. For the purposes of this paper, the general allowance list problem will be defined as the problem of properly stocking a naval vessel with the requisite supplies for a stated time period. The consolidation problem arises when an allowance list is prepared under the assumption that all the commodities listed in it are distinct, and later it is discovered that several of these commodities are, in fact, identical. Thus, for example, the allowance list may have a section dealing with replacement parts for electronic equipment. Suppose, for the sake of argument, that spare parts are required both for an amplifier and for an oscillator. It may turn out that both equipments require a certain spare tube, say, the 6L6 tube. The general allowance list had been constructed under the impression that the spare parts for the oscillator were entirely different from those used in the amplifier. Upon the discovery that the 6L6 tube occurs in both equipments, it then becomes advantageous to modify the allowance list by consolidating all the requirements for the 6L6 tube. This problem of the modification of the allowance list will be referred to as the consolidation problem. Its solution can often lead to substantial savings by reducing the quantity of replacement parts which should be carried aboard ship, where space is at a premium, and by increasing the probability that demands for the given part will be met.

In order to have a rational basis for the modification of an allowance list, it is necessary to understand the reasoning which underlies the construction of the original allowance list. This is still a largely empirical field, and many of the decisions entailed in the construction of an allowance list are necessarily arbitrary. In order to provide reasonable criteria for the modification of an existing allowance list, we shall assume that the general

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allowance list problem has been solved in accordance with certain general principles, and we shall then inquire into the modification of these rules made necessary by the discovery of the identity of some of the items. A rather idealized set of such rules has been proposed by the author in a previous article.[1] For the purposes of this paper, it is more appropriate to start afresh and develop a new treatment of the general allowance list problem, which will be particularly suitable for the deduction of simple, convenient rules of thumb for effecting a consolidation of the allowance list. We shall not aim for a mathematically complete exposition, but will strive for a plausible, reasonable and practical basis for the modification of existing procedures.

2. THE GENERAL ALLOWANCE LIST MODEL

As we mentioned earlier, the general allowance list is a listing of the quantities of the various commodities which should be carried aboard ship for a set time period. In arriving at a rational basis for the construction of such a list, we must decide upon the objectives which we shall strive to attain. In analogous economic problems, it is often customary to think of the basic objective of all economic activity, namely, the attainment of the greatest gain or, what is identical, the minimization of the loss. An analogous objective can be assumed for our problem with one important change—we must attribute dollar value to military accomplishment. It might seem inappropriate to measure the achievement of military operations in terms of dollars and cents. But whenever it is essential to be able to compare an economic quantity with a noneconomic quantity, the two quantities must be measured in the same units. This is the case, for example, when national security is assigned a certain dollar value in order to determine its share of the budget.

We can then set up a gain function, which represents the total gain, measured in dollars, achieved by the success of a logistics operation. This gain consists of a certain "satisfaction" achieved by fulfilling given demands, minus the cost of all the commodities stocked, and minus the cost of their storage space aboard ship. This "satisfaction" could be the accomplishment of a given military mission, or the support of a certain task force, or the attainment of any other predetermined military objective.

Thus, we can write our gain function as follows:

$$G_n(y_1, \ldots, y_n) = \sum_{\alpha=1}^{n} w_\alpha u_\alpha(y_\alpha) - \sum_{\alpha=1}^{n} s_\alpha y_\alpha - F\left(\sum_{\alpha=1}^{n} c_\alpha y_\alpha\right). \tag{2.1}$$

The terms in the gain function have the following significance:

The first term, $\sum_{\alpha} w_\alpha u_\alpha(y_\alpha)$, is the "satisfaction" achieved by fulfilling certain demands. Here w_α is the "military worth," in dollars, to the Navy of satisfying the demand for one unit of commodity α. The average quantity of commodity α delivered upon demand when an amount y_α of this commodity is stocked is represented by $u_\alpha(y_\alpha)$.

The second term, $\sum_{\alpha} s_\alpha y_\alpha$, is the total cost of all the commodities which are stocked aboard ship. Here, s_α is the cost, in dollars, of one unit of commodity α. Thus, $s_\alpha y_\alpha$ is

[1] M. H. Gourary, "An Optimum Allowance List Model," Naval Research Logistics Quarterly, Vol. 3 (1956), p. 177.

the cost of stocking an amount y_α of commodity α. For simplicity, it is assumed that this cost is a linear function of the quantity.

The last term, $F\left(\sum_\alpha c_\alpha y_\alpha\right)$, is some function giving the cost of occupied storage space aboard ship in dollars; and c_α is the number of cubic feet required for the storage of one unit of commodity α aboard ship. Thus, $\sum_\alpha c_\alpha y_\alpha$ is the total storage space, in cubic feet, occupied by all the commodities on the allowance list. Clearly, F must be a positive, monotonically increasing function of its argument, i.e., $F > 0$ and $F' \geq 0$.

The assignment of a cost function to space implies that space is at a premium aboard ship and that, for example, if the total space assigned to storage of material can be reduced, it can be utilized for other purposes.

Our goal in constructing the allowance list (i.e., in choosing $y_1, y_2, \ldots, y_n$) is to maximize the gain function $G_n(y_1, \ldots, y_n)$. Mathematically, this is accomplished by differentiating $G_n(y_1, \ldots, y_n)$ with respect to the y_α's and setting the n partial derivatives equal to zero.

$$(2.2) \qquad \frac{\partial G_n(y_1, \ldots, y_n)}{\partial y_\alpha} = 0 = w_\alpha \frac{du_\alpha(y_\alpha)}{dy_\alpha} - S_\alpha - c_\alpha F'\left(\sum_\alpha c_\alpha y_\alpha\right)$$

$$\text{for } \alpha = 1, \ldots, n .$$

By definition, $u_\alpha(y_\alpha)$ is the average quantity of commodity α which is delivered upon demand when an amount y_α of that commodity is stocked. Let x_α be the total amount of commodity α demanded during the fixed time period for which this allowance list is constructed. If $\phi_\alpha(x_\alpha)\,dx_\alpha$ is the probability that the total amount demanded of commodity α has a value between x_α and $x_\alpha + dx_\alpha$, then $\int_0^{y_\alpha} x_\alpha \phi_\alpha(x_\alpha)\,dx_\alpha$ is the average total amount of commodity α, supplied in response to demands for a total amount of this commodity which is less than the amount stocked, y_α. Clearly, such demands are always fulfilled. The probability that the total amount demanded of commodity α exceeds the amount stocked is given by $\int_{y_\alpha}^{\infty} \phi_\alpha(x_\alpha)\,dx_\alpha$. In this case we supply only an amount y_α. Thus, the average total amount supplied when the demand x_α exceeds the supply y_α is $y_\alpha \int_{y_\alpha}^{\infty} \phi_\alpha(x_\alpha)\,dx_\alpha$. The total amount supplied on the average when an amount y_α is stocked is, therefore,

$$(2.3) \qquad u_\alpha(y_\alpha) = \int_0^{y_\alpha} x_\alpha \phi_\alpha(x_\alpha)\,dx_\alpha + y_\alpha \int_{y_\alpha}^{\infty} \phi_\alpha(x_\alpha)\,dx_\alpha .$$

Now,

$$\frac{du_\alpha(y_\alpha)}{dy_\alpha} = y_\alpha \phi_\alpha(y_\alpha) - y_\alpha \phi_\alpha(y_\alpha) + \int_{y_\alpha}^{\infty} \phi_\alpha(x_\alpha)\, dx_\alpha ,$$

$$= \int_{y_\alpha}^{\infty} \phi_\alpha(x_\alpha)\, dx_\alpha .$$

Thus the allowance list equations (2.2) become:

$$\int_{y_\alpha}^{\infty} \phi_\alpha(x_\alpha)\, dx_\alpha = \frac{s_\alpha}{w_\alpha} + \frac{c_\alpha}{w_\alpha} F'\left(\sum_\alpha c_\alpha y_\alpha\right), \qquad \alpha = 1, \ldots, n . \tag{2.4}$$

For a certain class of distributions, $\phi_\alpha(x_\alpha)$, and a certain class of functions, F, this set of equations uniquely determines the set of values of y_α's which actually maximize the gain function. We shall not investigate the properties of these classes of functions, but will confine ourselves to a study of a number of numerical models which will enable us to acquire good judgment and convenient rules of thumb for the treatment of practical problems.

The set of simultaneous equations (2.4) can often be solved by an iteration procedure. This consists of first assuming a certain numerical value for $C \equiv \sum_\alpha c_\alpha y_\alpha$, computing the y_α's from (2.4), and then recomputing C. This procedure is repeated as many times as necessary for the final C to agree with the preceding one. This procedure converges rather rapidly in certain cases. We shall not investigate its convergence for the general case, but will instead proceed to the study of several numerical examples.

2.1 Numerical Model

In this model, we shall assume that the $\phi_\alpha(x_\alpha)$ are normal distributions and we shall choose an exponential for the function F. These are reasonable choices because the normal distribution is a limiting form for a large number of practically significant distributions, such as the Poisson distribution. The rising exponential also seems to be a desirable choice for F because it satisfies the two conditions $F > 0$ and $F' \geq 0$. It rises slowly at first, and then its rise becomes steeper. This is a reasonable behavior for a function representing the cost of space aboard ship. It is also a rather smooth function, and this tends to improve the convergence of our iteration procedure. Symbolically, then, $F(C) = Ae^{bC}$, where $A > 0$ and $b > 0$.

Since $$\phi_\alpha(x_\alpha) = N(m_\alpha, \sigma_\alpha^2) = \frac{1}{\sqrt{2\pi}\,\sigma_\alpha} e^{-\frac{1}{2}\left(\frac{x_\alpha - m_\alpha}{\sigma_\alpha}\right)^2},$$

the allowance list equations (2.4) become:

$$\frac{1}{\sqrt{2\pi}\,\sigma_\alpha}\int_{y_\alpha}^{\infty} e^{-\frac{1}{2}\left(\frac{x_\alpha - m_\alpha}{\sigma_\alpha}\right)^2} dx_\alpha = \frac{s_\alpha}{w_\alpha} + \frac{c_\alpha}{w_\alpha} F'\left(\sum_\alpha c_\alpha y_\alpha\right);$$

or, by changing to the standardized variables $z_\alpha = \dfrac{y_\alpha - m_\alpha}{\sigma_\alpha}$, we get:

$$\text{(2.5)} \qquad \frac{1}{\sqrt{2\pi}}\int_{z_\alpha}^{\infty} e^{-\frac{1}{2}t^2} dt = \frac{s_\alpha}{w_\alpha} + \frac{c_\alpha}{w_\alpha} F'\left(\sum_\alpha c_\alpha y_\alpha\right); \quad \alpha = 1, \ldots, n.$$

With the allowance list equations in this form, z_α can be easily determined from tables of normal probability functions.

Evidently, z_α depends only on the individual $\dfrac{s_\alpha}{w_\alpha}, \dfrac{c_\alpha}{w_\alpha}$, and on a property of the whole allowance list, namely, $F'\left(\sum_\alpha c_\alpha y_\alpha\right)$. Thus, if items 1 and 2 have

$$\frac{s_1}{w_1} = \frac{s_2}{w_2} \quad \text{and} \quad \frac{c_1}{w_1} = \frac{c_2}{w_2},$$

then $z_1 = z_2$. We shall find this fact useful when we tackle the consolidation of the allowance list.

Let us consider the numerical model of Table 1. This model has been chosen with various combinations of mean, variance, price, cube, and "military worth" in order for it to contain some elements of reality. With the choice of the rising exponential F, we shall consider several combinations of the constants A and b. Later we shall discuss the significance of these choices.

2.2 Example 1

Here we set $b = 0$; therefore, $F = A$. This means that the cost of storage space aboard ship is considered to be fixed overhead. Thus, the y_α's are independent of cost of storage space. They depend only on the cost of the items themselves, s_α, and on their "military worth," w_α. Then, the amount stocked of item 1, y_1, is independent of the other amounts stocked. This is a simple case and will serve to introduce us to the various problems involved.

The y_α's, the $u_\alpha(y_\alpha)$'s, and $z_\alpha = \dfrac{y_\alpha - m_\alpha}{\sigma_\alpha}$ appear in Table 2. We define P_α as the percentage of $\left(\dfrac{\text{average quantity supplied}}{\text{average quantity demanded}}\right)$. Symbolically, $P_\alpha = 100\dfrac{u_\alpha(y_\alpha)}{m_\alpha}$. Clearly, $P_\alpha \leq 100$, because the average quantity supplied upon demand can never exceed the average quantity demanded. For this model the cube stocked is $C = \sum_\alpha c_\alpha y_\alpha = 2359.1$. The gain function attains the value $G_{24} = 15{,}786.6$. For simplicity in the ensuing discussion we shall set $A = 0$.

TABLE 1

Group	α	m_α	σ_α	w_α	s_α	c_α
a	1	20	6.7	5	1	1
	2	20	4.0	5	1	1
	3	40	13.3	5	1	1
	4	40	8.0	5	1	1
b	5	20	6.7	40	1	5
	6	20	4.0	40	1	5
	7	40	13.3	40	1	5
	8	40	8.0	40	1	5
c	9	20	6.7	10	1	1
	10	20	4.0	10	1	1
	11	40	13.3	10	1	1
	12	40	8.0	10	1	1
d	13	20	6.7	10	1	2
	14	20	4.0	10	1	2
	15	40	13.3	10	1	2
	16	40	8.0	10	1	2
e	17	20	6.7	50	10	1
	18	20	4.0	50	10	1
	19	40	13.3	50	10	1
	20	40	8.0	50	10	1
f	21	20	6.7	50	10	5
	22	20	4.0	50	10	5
	23	40	13.3	50	10	5
	24	40	8.0	50	10	5

TABLE 2

Group	α	z_α	y_α	$u_\alpha(y_\alpha)$	P_α
a	1	0.8415	25.6	19.2	96.0
	2	0.8415	23.4	19.6	98.0
	3	0.8415	51.2	38.5	96.3
	4	0.8415	46.7	39.1	97.8
b	5	1.960	33.1	19.9	99.5
	6	1.960	27.8	19.9	99.8
	7	1.960	66.1	39.9	99.7
	8	1.960	55.7	39.9	99.7
c	9	1.281	28.6	19.7	98.5
	10	1.281	25.1	19.8	99.0
	11	1.281	57.0	39.4	98.5
	12	1.281	50.2	39.6	99.0
d	13	1.281	28.6	19.7	98.5
	14	1.281	25.1	19.8	99.0
	15	1.281	57.0	39.4	98.5
	16	1.281	50.2	39.6	99.0
e	17	0.8415	25.6	19.2	96.0
	18	0.8415	23.4	19.6	98.0
	19	0.8415	51.2	38.5	96.3
	20	0.8415	46.7	39.1	97.8
f	21	0.8415	25.6	19.2	96.0
	22	0.8415	23.4	19.6	98.0
	23	0.8415	51.2	38.5	96.3
	24	0.8415	46.7	39.1	97.8

2.3 Example 2

Here $b = \frac{1}{1000}$ and $A = 200$. This means that the y_α's depend on the unit cost, the "military worth," and also on the cost of storage space aboard ship. Thus the y_α's are interdependent, and must be determined by an iteration of the allowance list equations (2.5). Table 3 gives the y_α's, $u_\alpha(y_\alpha)$'s and P_α's, as well as $z_\alpha = \frac{y_\alpha - m_\alpha}{\sigma_\alpha}$.

For the first step in the iteration we choose $C = 2000$, somewhat larger than $\sum_\alpha c_\alpha m_\alpha = 1800$. This leads to $\sum_\alpha c_\alpha y_\alpha = 2054.5$. Next we take $C = 2030$ and get $\sum_\alpha c_\alpha y_\alpha = 2046.4$. Finally, we take $C = 2040$ and obtain $\sum_\alpha c_\alpha y_\alpha = 2044.2$; after two iterations our process converges. The gain function, G_{24}, has the value 14,010.8.

2.4 Example 3

In this example, we shall set $s_\alpha = 0$ for all α; $b = \frac{1}{1000}$ and $A = 200$. This means that the cost of the individual commodities plays no role in the construction of the allowance list. The results are given in Table 4. The computations of this table required one iteration. The final value of $\sum_\alpha c_\alpha y_\alpha = 2190.4$, and the gain function $G_{24} = 17{,}505.0$.

TABLE 3

Group	α	z_α	y_α	$u_\alpha(y_\alpha)$	P_α
a	1	-0.0190	19.9	17.3	86.5
	2	-0.0190	20.0	18.4	92.0
	3	-0.0190	39.7	34.5	86.2
	4	-0.0190	39.9	36.8	92.0
b	5	0.7817	25.2	19.2	96.0
	6	0.7817	23.1	19.5	97.5
	7	0.7817	50.4	38.4	96.0
	8	0.7817	46.3	39.0	97.5
c	9	0.6626	24.4	19.0	95.0
	10	0.6626	22.7	19.4	97.0
	11	0.6626	48.8	38.0	95.0
	12	0.6626	45.3	38.8	97.0
d	13	0.2337	21.6	18.1	90.5
	14	0.2337	20.9	18.8	94.0
	15	0.2337	43.1	36.1	90.3
	16	0.2337	41.9	37.6	94.0
e	17	0.7362	24.9	19.1	95.5
	18	0.7362	22.9	19.5	97.5
	19	0.7362	49.8	38.2	95.5
	20	0.7362	45.9	38.9	97.2
f	21	0.3751	22.5	18.4	92.0
	22	0.3751	21.5	19.0	95.0
	23	0.3751	45.0	36.8	92.0
	24	0.3751	43.0	38.1	95.2

TABLE 4

Group	α	z_α	y_α	$u_\alpha(y_\alpha)$	P_α
a	1	0.3654	22.4	18.4	92.0
	2	0.3654	21.5	19.0	95.0
	3	0.3654	44.8	36.8	92.0
	4	0.3654	42.9	38.0	95.0
b	5	0.7607	25.1	19.1	95.5
	6	0.7607	23.0	19.5	97.5
	7	0.7607	50.1	38.2	95.5
	8	0.7607	46.1	39.0	97.5
c	9	0.9203	26.1	19.3	96.5
	10	0.9203	23.7	19.6	98.0
	11	0.9203	52.2	38.6	96.5
	12	0.9203	47.4	39.2	98.0
d	13	0.3654	22.4	18.4	92.0
	14	0.3654	21.5	19.0	95.0
	15	0.3654	44.8	36.8	92.0
	16	0.3654	42.9	38.0	95.0
e	17	1.803	32.1	19.9	99.5
	18	1.803	27.2	19.9	99.5
	19	1.803	64.0	39.8	99.5
	20	1.803	54.4	39.9	99.8
f	21	0.9203	26.1	19.3	96.5
	22	0.9203	23.7	19.6	98.0
	23	0.9203	52.2	38.6	96.5
	24	0.9203	47.4	39.2	98.0

2.5 Discussion of the General Allowance List

A perusal of Tables 2 and 3 shows certain characteristic results. We note that the only difference between Examples 1 and 2 is that in Example 1 $F = A =$ constant, which may be zero, while in Example 2, $F = 200e^{C/1000}$. In other words, cost of space aboard ship is constant in the first example, and it is an exponentially rising function of the total cube stocked in

the second example. This leads to smaller y_α in the second example. Not all the y_α's decrease to the same extent in going from Example 1 to Example 2.

It is most convenient to measure the decrease in terms of P_α. The greatest decrease occurs for $\alpha = 1, 3, 13$, and 15. These commodities have large σ_α, small w_α, and relatively large $\frac{c_\alpha}{w_\alpha}$. The large σ_α means that the future demand for these commodities is rather uncertain. The small w_α implies that these commodities are not of great importance. And the relatively large $\frac{c_\alpha}{w_\alpha}$ means that they have a relatively large cube for their "military worth." Thus, the space cost has a greater effect on these four commodities.

The smallest changes occur in Group e, because these commodities have so small a cube in relation to their "military worth" that the space constraint does not affect them much.

As we go from Example 2 to Example 3, the P_α's increase again. The only difference between these two examples is that in Example 3 we have set $s_\alpha = 0$ for all α without altering the cost of storage space. In Group b, which has the smallest $\frac{s_\alpha}{w_\alpha}$, the change is negligibly small. (Because of round-off error, two of the P_α's in Group b actually appear to decrease by 1/2 percent in going from Example 2 to Example 3.) In Groups a, c, and f, where $\frac{s_\alpha}{w_\alpha}$ is large, the increase in P_α is sizeable. For equal $\frac{s_\alpha}{w_\alpha}$, the increase in P_α in going from Example 2 to Example 3 is most pronounced for the smaller values of P_α in Example 2. This fact indicates that when the unit price is lowered, our allowance list equations require that the severe shortages be eliminated first.

3. CONSOLIDATED ALLOWANCE LIST MODEL

Whenever we discover that several of the items on the allowance list are identical, we have a consolidation problem on our hands. If we find that all the items in Group a of Table 1, i.e., items 1 through 4, are identical, then we want to consolidate them. For the four commodities to be identical, it is necessary that their price and cube, s_α and c_α, be identical; that is, $s_1 = s_2 = s_3 = s_4$ and $c_1 = c_2 = c_3 = c_4$. In principle, the w_α's need not be equal, because a replacement part for one type of apparatus may be identical with that of another type of apparatus, although its "military worth" is very different from the "military worth" of the first one. Since unequal w_α's would complicate the consolidation problem, we shall confine our present discussion to the case of equal w_α's, i.e., $w_1 = w_2 = w_3 = w_4$.

If the original distribution functions of the demands x_1, x_2, x_3, and x_4 are:

$$\phi_1(x_1) = N(m_1, \sigma_1^2)^* \qquad \phi_3(x_3) = N(m_3, \sigma_3^2)$$

$$\phi_2(x_2) = N(m_2, \sigma_2^2) \qquad \phi_4(x_4) = N(m_4, \sigma_4^2)$$

---

*$N(m_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2\sigma_i^2}(x_i - m_i)^2}$

then, we define the aggregate demand for Group a as:

$$x_a = x_1 + x_2 + x_3 + x_4 .$$

Clearly, $$m_a = m_1 + m_2 + m_3 + m_4 ,$$

and $$\sigma_a^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 .$$

The distribution function of the aggregate demand is then:

(3.1) $$\phi_a(x_a) = N(m_a, \sigma_a^2) .$$

Therefore the new gain function for the consolidated allowance list is:

$$G_{21}^{(a)}(y_a, y_5, \ldots, y_{24}) = \sum_{\alpha=5}^{24} w_\alpha u_\alpha(y_\alpha) + w_a u_a(y_a) - \sum_{\alpha=5}^{24} s_\alpha y_\alpha - s_a y_a - F\left(\sum_{\alpha=5}^{24} c_\alpha y_\alpha + c_a y_a\right),$$

where we must remember that

$$u_a(y_a) = \int_0^{y_a} x_a \phi_a(x_a)\, dx_a + y_a \int_{y_a}^{\infty} \phi_a(x_a)\, dx_a .$$

[$\phi_a(x_a)$ now appears instead of $\phi_1(x_1)$, ... etc.; $c_a = c_1 = c_2 = c_3 = c_4$; $s_a = s_1 = s_2 = s_3 = s_4$; and $w_a = w_1 = w_2 = w_3 = w_4$.]

In principle, we should now proceed to maximize $G_{21}^{(a)}$ with respect to the twenty-one y_α's. In practice, this is usually neither desirable nor necessary, because the consolidation of several items in a large allowance list will hardly affect the remainder of the allowance list. We shall assume, therefore, that $y_5, \ldots, y_{24}$ in the consolidated allowance list have the same values as in the original allowance list (Table 2 or Table 3), and we proceed to maximize $G_{21}^{(a)}$ with respect to the only remaining variable, namely, y_a. This gives us the one required equation, namely,

$$\frac{\partial G_{21}^{(a)}(y_a, y_5, \ldots, y_{24})}{\partial y_a} = 0 = w_a \frac{du_a(y_a)}{dy_a} - s_a - c_a F'\left(\sum_{\alpha=5}^{24} c_\alpha y_\alpha + c_a y_a\right)$$

or

(3.2) $$\int_{y_a}^{\infty} \phi_a(x_a)\, dx_a = \frac{s_a}{w_a} + \frac{c_a}{w_a} F'\left(\sum_{\alpha=5}^{24} c_\alpha y_\alpha + c_a y_a\right).$$

The solution of this equation is obtained by iteration. In practice, it will be desirable to have a convenient rule for guessing at the solution of this equation. Such a rule can easily be given for the case of normally distributed $x_1, \ldots, x_4$. Namely, if $y_1 = m_1 + z_1 \sigma_1$, then

(3.3) $$y_a = m_a + z_a\sigma_a \text{, where } z_a = z_1 = z_2 = z_3 = z_4 .$$

We recall that $z_1, \ldots, z_4$ were determined for the original allowance list before we considered consolidating it. They were obtained from equations (2.5) which are:

(2.5) $$\frac{1}{\sqrt{2\pi}} \int_{z_\alpha}^{\infty} e^{-\frac{t^2}{2}}\, dt = \frac{s_\alpha}{w_\alpha} + \frac{c_\alpha}{w_\alpha} F'\left(\sum_{\alpha=1}^{24} c_\alpha y_\alpha\right), \quad \alpha = 1, \ldots, 4 .$$

On the other hand, z_a is determined from an analogous equation obtained by substituting the normal distribution function (3.1) into the consolidated allowance list equation (3.2). The resulting equation for z_a is:

(3.4) $$\frac{1}{\sqrt{2\pi}} \int_{z_a}^{\infty} e^{-\frac{t^2}{2}}\, dt = \frac{s_a}{w_a} + \frac{c_a}{w_a} F'\left(\sum_{\alpha=5}^{24} c_\alpha y_\alpha + c_a y_a\right).$$

Let us compare the individual terms on the right-hand side of (3.4) with the corresponding terms of (2.5) for, say, $\alpha = 2$. (We have picked a value of α, corresponding to a commodity in Group a.) Then, $\frac{s_a}{w_a} = \frac{s_2}{w_2}$ and $\frac{c_a}{w_a} = \frac{c_2}{w_2}$. The only term that is different in the two expressions is F'. Fortunately, it turns out that F', though different in appearance, is essentially unchanged in numerical value in the two equations. Therefore, $z_a = z_2$ to a good approximation.

The insensitivity of F' to the consolidation of several commodities will be brought out in the numerical examples. It is not, however, limited to the specific numerical model that we have considered. This can be seen as follows. $F(C)$ is the cost of space aboard ship as a function of total cube. It can be expected to have the general form of the curve drawn in Figure 1. $F'(C)$ is the slope of this curve (which is the rate of change of the cost with cube) at the point C. Our problem entails the approximation of $F'(C_{\text{consolidated A.L.}})$ by $F'(C_{\text{original A.L.}})$. Since the total cube of the two allowance lists will rarely change by more than several percent, $F'(C)$ is very likely to remain virtually unchanged.

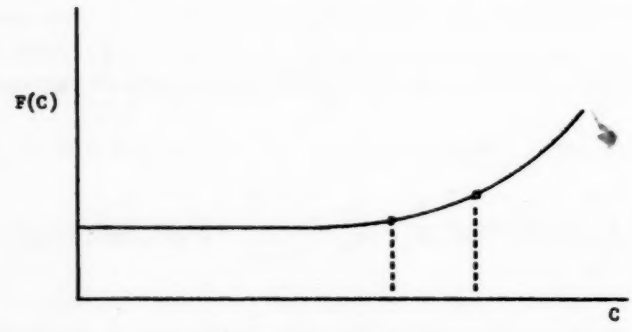

Figure 1

In a later section, we shall propose that formula (3.3) be used as a general rule of thumb for the consolidation of most allowance lists, and we shall attempt to delimit its domain of applicability. Before we do this, however, we must study carefully the following numerical examples.

It should be clear, of course, that while we have talked about the consolidation of the items in Group a, an analogous argument is valid for the consolidation of the items in any one of the other five groups. The numerical examples will, in fact, list the results obtained in all these consolidations. It is also possible, of course, to consolidate the items in Group a and simultaneously to consolidate the items in Group d, but we shall not perform these more elaborate computations.

3.1 Example 1

We shall turn now to the computation of the consolidated allowance list based on the model of the original allowance list which appears in Table 1. We shall divide the twenty-four commodities into six groups of four each (Groups a, b, c, d, e, and f) and proceed to consolidate each of these groups in turn. s_α, c_α, and w_α do not vary from item to item within each group. The results obtained through consolidation for each of the six groups appear in Table 5.

TABLE 5

Group i	i = a α = 1, 2, 3, 4	i = b α = 5, 6, 7, 8	i = c α = 9, 10, 11, 12	i = d α = 13, 14, 15, 16	i = e α = 17, 18, 19, 20	i = f α = 21, 22, 23, 24
Y_i	146.9	182.7	160.9	160.9	146.9	146.9
y_i	134.7	154.2	144.3	144.3	134.7	134.7
P_i	96.0 - 98.0	99.5 - 99.8	98.5 - 99.0	98.5 - 99.0	96.0 - 98.0	96.0 - 98.0
p_i	98.4	99.83	99.5	99.5	98.4	98.4

(Note that the amounts stocked of items not consolidated are kept the same as in the original allowance list.) In this example F = A and F' = 0. Y_i is defined as the sum of the four y_α's in Group i (i = a, b, . . . , f) appearing in the original allowance list of Table 2. It is, therefore, the total amount of the commodity, a, which was to have been stocked originally. On the other hand, y_i is the amount of commodity, a, which is to be stocked according to our present consolidation scheme. P_i is the range of the P_α's in Group i; and p_i is the percentage of $\left(\frac{\text{average quantity supplied}}{\text{average quantity demanded}}\right)$ for the consolidated allowance list.

A brief look at this table convinces us that the consolidation has not decreased P_i, that is, $p_i \geq P_i$. Roughly speaking, this means that the probability of meeting the demands has certainly not decreased with the smaller consolidated y_i. In fact, because of the consolidation, which means stocking the four items in one batch, the probability of meeting the demands is often larger.

Let us next examine how the consolidation has affected the maximum of the gain function. Originally, the allowance list led to G_{24}. The consolidated allowance list leads to $G_{21}^{(i)}$, i = a, ..., f. In order to provide a measure of the percentage increase in the gain due to the consolidation, we introduce the measure

$$g^{(i)} = \frac{G_{21}^{(i)} - G_{24}}{G_{24} - G_{20}^{(i)}}$$

and will call it the relative consolidation gain of the i-th group. Here $G_{20}^{(i)}$ is defined as the maximum value of the gain function when the commodity of Group i is deleted from the allowance list altogether. Thus, $G_{20}^{(i)}$ is the maximum value of the gain function for an original allowance list consisting of twenty commodities. It follows that $G_{24} - G_{20}^{(i)}$ is the gain achieved by including the commodity of Group i in the original allowance list and that $G_{21}^{(i)} - G_{24}$ is the gain due to consolidation. Thus,

$$g^{(i)} = \left(\frac{\text{gain due to consolidation of Group i}}{\text{gain due to inclusion of Group i}}\right).$$

These quantities are listed in Table 6. A detailed discussion of Tables 4 and 5 will follow Example 3.

TABLE 6

Group i	i = a $\alpha = 1, \ldots, 4$	i = b $\alpha = 5, \ldots, 8$	i = c $\alpha = 9, \ldots, 12$	i = d $\alpha = 13, \ldots, 16$	i = e $\alpha = 17, \ldots, 20$	i = f $\alpha = 21, \ldots, 24$
G_{24}	15,786.6	15,786.6	15,786.6	15,786.6	15,786.6	15,786.6
$G_{21}^{(i)}$	15,807.2	15,823.1	15,812.2	15,812.2	15,993.6	15,993.6
$G_{20}^{(i)}$	15,351.2	11,185.3	14,762.5	14,762.5	11,435.6	11,435.6
$g^{(i)}$	4.7%	0.8%	2.5%	2.5%	4.7%	4.7%

3.2 Example 2

Here $F = 200\, e^{C/1000}$ and $F' = .2\, e^{C/1000}$. Since F' is not zero in this example, it will provide us with a numerical test of our proposed consolidation rule, namely, $y_i = m_i + z_i \sigma_i$, where z_i is set equal to one of the z_α's obtained originally for the i-th group (of the original allowance list). We will tabulate both $y_i^{(1)}$, which denotes the value of y_i obtained from the proposed consolidation rule, and y_i, which denotes the final value after the required iterations. It is evident that our rule works extremely well in this particular example. Table 7 lists these results. The corresponding gain functions are given in Table 8.

TABLE 7

Group i	i = a $\alpha = 1, \ldots, 4$	i = b $\alpha = 5, \ldots, 8$	i = c $\alpha = 9, \ldots, 12$	i = d $\alpha = 13, \ldots, 16$	i = e $\alpha = 17, \ldots, 20$	i = f $\alpha = 21, \ldots, 24$
Y_i	119.5	145.0	141.2	127.5	143.5	132.0
$y_i^{(1)}$	119.7	133.6	131.5	124.1	132.8	126.5
y_i	119.7	134.3	131.6	124.1	132.8	126.5
P_i	86.0 - 92.0	96.0 - 97.5	95.0 - 97.0	90.0 - 94.0	95.5 - 97.5	92.0 - 95.0
p_i	94.1	98.3	97.8	95.8	98.1	96.5

TABLE 8

Group i	i = a $\alpha = 1, \ldots, 4$	i = b $\alpha = 5, \ldots, 8$	i = c $\alpha = 9, \ldots, 12$	i = d $\alpha = 13, \ldots, 16$	i = e $\alpha = 17, \ldots, 20$	i = f $\alpha = 21, \ldots, 24$
G_{24}	14,010.8	14,010.8	14,010.8	14,010.8	14,010.8	14,010.8
$G_{21}^{(i)}$	14,040.1	14,194.5	14,052.4	14,061.2	14,227.8	14,240.8
$G_{20}^{(i)}$	13,762.0	10,355.0	13,234.0	13,379.5	12,447.9	10,483.9
$g^{(i)}$	11.8%	5.0%	5.4%	8.0%	13.9%	6.5%

3.3 Example 3

In this example all $s_\alpha = 0$, $F = 200\, e^{C/1000}$, and $F' = .2\, e^{C/1000}$. This case confronts our proposed consolidation rule with its severest test yet, because here only the cost of storage space limits the stocks. Table 9 shows that our rule works very well indeed even under these circumstances. The corresponding gain functions are listed in Table 10.

3.4 Discussion of the Consolidated Allowance List

Three facts emerge from the above examples. First, the consolidation leads to a substantial relative consolidation gain, $g^{(i)}$, sometimes as large as 14 percent. Second, the proportion of the demand met, P_α, is never decreased by the consolidation, and it is often increased by several percent. In fact, if a group of commodities are consolidated, then the resulting p_i will be at least as large as the largest of the original P_α's and frequently somewhat larger. And third, our proposed consolidation rule works very well under a variety of conditions.

The reason for the increase in P_α is not hard to understand. When the several consolidated commodities are stocked in one place, the proportion of demand met is bound to increase. This in turn contributes to an increase in the gain.

TABLE 9

Group i	i = a $\alpha = 1, \ldots, 4$	i = b $\alpha = 5, \ldots, 8$	i = c $\alpha = 9, \ldots, 12$	i = d $\alpha = 13, \ldots, 16$	i = e $\alpha = 17, \ldots, 20$	i = f $\alpha = 21, \ldots, 24$
Y_i	131.6	144.3	149.4	131.6	177.7	149.4
$y_i^{(1)}$	126.4	133.2	136.0	126.4	151.4	136.0
y_i	126.4	133.9	136.2	126.5	151.6	136.8
P_i	92.0 - 95.0	95.5 - 97.5	96.5 - 98.0	92.0 - 95.0	99.5 - 99.8	96.5 - 98.0
p_i	96.5	98.3	98.8	96.5	99.83	98.8

TABLE 10

Group i	i = a $\alpha = 1, \ldots, 4$	i = b $\alpha = 5, \ldots, 8$	i = c $\alpha = 9, \ldots, 12$	i = d $\alpha = 13, \ldots, 16$	i = e $\alpha = 17, \ldots, 20$	i = f $\alpha = 21, \ldots, 24$
G_{24}	17,505.0	17,505.0	17,505.0	17,505.0	17,505.0	17,505.0
$G_{21}^{(i)}$	17,532.0	17,685.0	17,545.0	17,559.0	17,566.0	17,710.0
$G_{20}^{(i)}$	17,138.0	13,859.0	16,574.5	16,572.0	11,869.5	11,882.5
$g^{(i)}$	7.4%	4.9%	4.3%	5.8%	1.1%	3.6%

The validity of our consolidation rule is, practically speaking, our most important result. It is simple enough so that it can be applied, even though the original allowance list may have been constructed by a method very different from the one described here. It requires only the knowledge of the first and second moments of the distributions of demand. It also does not depend on the explicit knowledge of the w_α's. While we have thus far assumed normality, we will show in Section 4 that analogous consolidation rules can be stated for other distributions as well.

4. GENERALIZATION TO OTHER DISTRIBUTIONS

The consolidation rule has been stated so far only for normally distributed demands. It will now be formulated for other distributions. It states that in the consolidated allowance-list equation (3.2) the right-hand side may be approximated by the right-hand side of the original allowance-list equations (2.4) for one of the corresponding values of α. We must now obtain $\phi_a(x_a)$ from the $\phi_\alpha(x_\alpha)$'s by the convolution method.[2] This is obviously useless except for those distributions which reproduce themselves under addition. Even then, however, it requires more computations than in the normal case.

[2]See, for example, Harald Cramer, "Mathematical Methods of Statistics," Princeton University Press (1946), p. 196.

5. CONCLUSIONS

The present state of the art is such that allowance lists are constructed without the benefit of complete and detailed information about the nature of demand distributions. It is to be hoped that as the science of logistics continues its growth and development, and as more usage data become available, the construction of allowance lists may be put on a firmer foundation. It appears, however, that the consolidation problem can be treated adequately, even at this early stage.

Since our present knowledge of the demand distributions is very incomplete, it is fairly reasonable and practically advantageous to assume normality. The consolidation rule for the normal case is simple enough to be applied. Because of the Central Limit Theorem[3] the consolidation rule in its normal form will be correct in many cases. There will be a few situations where this rule will fail. But a rule which is simple to use and which is correct in the great majority of cases is all that can be desired until more information about demand distributions becomes available.

6. ACKNOWLEDGMENTS

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[3] Ibid., p. 214.

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A SELECTED, DESCRIPTIVE BIBLIOGRAPHY OF REFERENCES ON PRIORITY SYSTEMS AND RELATED, NONPRICE ALLOCATORS†

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Among the problems encountered in the design of an optimal military supply system, the question of an efficient priority or allocation system looms as one of major importance. This has been generally recognized by serious students of military supply, since the breakdown of priority systems has proved a major source of difficulty to the military in past emergencies.[1]

The general priority question and the related problems of allocations and limitational orders are, however, of a somewhat greater scope than merely their application to the control of military supply or war production: they are, in fact, central to a number of related problems in theoretical and applied economics. While the solution of the problem of efficient allocation is economics in its broadest sense, that science has concentrated a disproportionate share of its attention on the allocating function of the price system. In many instances, however — for example, in the operation of a Socialist or centrally-controlled economy, in consumer rationing, in production scheduling, or in the choice between alternative investment plans within a firm — the allocating function of the price system is either ineffective, or — the general case — is not completely superseded but must be performed in conjunction with other factors.[2] It is the last — the combinatorial case — which provides the greatest theoretical problem, and it is precisely in this case that our knowledge of the theory which underlies the operation of the system is most deficient.[3]

As an initial step in the study of the priority problem the following listing of references, with brief descriptions of their contents, has been prepared. Since, as is noted above, the priority question enters into many economic problems in different forms, it is our feeling that all aspects of the question should be considered in order that a truly general approach may be developed and the problem of military supply may benefit from theoretical developments in

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[1]We cite, for example, Rear Admiral Henry E. Eccles, USN (Ret.), Command Logistics, U.S. Naval War College, Newport (1956), pp. 110-111.

[2]The view that the price system is ineffective in certain situations, in particular in the case of rationing and control of war production, has not been accepted by all economists (cf. F. A. v. Hayek, "Pricing Versus Rationing," The Banker, Vol. 51, No. 164, Sept. 1939, pp. 242-249), but past experience has clearly demonstrated that no central authority would rely entirely on the price system in periods of emergency as the only control over distribution.

[3]". . .there exists no real theory of priorities; especially the working of priorities in systems where also prices prevail and localized decisions are frequently possible, is only imperfectly understood." Oskar Morgenstern, "Consistency Problems in the Military Supply System," Naval Research Logistics Quarterly, Vol. 1, No. 4, Dec. 1954, pp. 265-281.

other fields. The bibliography has been prepared with this in mind.[4] Section IIIB provides an example of how this more general approach may be of some value. Although the theory of queues or waiting lines has received considerable attention in recent years, there has been, to my knowledge, little or no application of this research in the literature produced by the numerous groups or persons interested in the theoretical problems of logistics, although some of these have also made contributions to waiting-line theory. In actuality, the problem of waiting lines is a quite common one in military supply: ships must wait for dry dock facilities, men wait for transport space, messages wait for access to communication lines. A special aspect of waiting lines which has been recently treated in the literature and which is especially relevant to problems of military supply deals with the question of priorities. It is highly unlikely that one message is of equal importance with another or that repairs on minor auxiliary vessels are of equal military importance with the repair of major combatant ships. Recognition of this fact involves the assignment of priorities to units in the line waiting for service. Any additional information or insights which might be contributed by interested readers will materially aid the progress of this study.

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[4]The question of space and the present limited scope of the study make it impossible to list all or even a major portion of the theoretical material which may be relevant to the priority problem. We confine ourselves here to material which deals with several aspects of priorities and allocations per se. On the closely related topic of consumer rationing, see James Tobin, "A Survey of the Theory of Rationing," Econometrica, Vol. 20, No. 4, Oct. 1952, pp. 521-553.

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A METHOD OF ESTIMATING SPARE-PART ESSENTIALITY[1]

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This paper contains the text of a briefing on a method of estimating spare-part "essentiality" or "shortage penalties." The estimates are made by collecting and combining the judgments of individual maintenance specialists. While the talk was primarily directed to a military audience, the subject appears to be of general interest, since the problem of estimating shortage penalties must be faced in applying any of the optimal inventory policies which have been developed in recent years. The quantitative solution of many other management problems also requires input data which can only be obtained on the basis of expert judgment.

INTRODUCTION

I would like to discuss with you a method which we have recently developed for estimating what is variously called spare-part "essentiality," "criticality," or "shortage penalty." As you all know, shortages of some parts are more serious than shortages of others; and if these differences can be taken account of in making procurement and stocking decisions, the usefulness of any given amount of inventory investment can be increased. The evaluation of supply management's performance also can be made more precise if the differences in shortage seriousness can be taken account of.

To lend perspective to the concept of "essentiality," I shall first illustrate briefly one of the ways in which these measures can be used by introducing them into the familiar flyaway-kit example.[2] Second, I shall discuss some of the major elements to be considered in estimating essentiality factors. Third, I shall describe an experiment which used psychometric scaling techniques to combine the judgment of several experienced maintenance men to categorize a sample of F-86H spare parts according to their essentiality. Finally, I shall discuss a rationale for assigning essentiality or shortage penalty factors to these categories of items.

USEFULNESS OF ESSENTIALITY FACTORS

Let us consider first the way in which essentiality factors can be used in the design of flyaway kits. (With slight modification, the same types of essentiality factors are equally

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[1]Originally presented at the Eighth Meeting of the Long Range Logistic Discussion Group, held at The RAND Corp., Santa Monica, Feb. 14, 1957. This material also appeared as Unclassified RAND paper P-1064, Apr. 17, 1957.

[2]A flyaway kit is a mobile package of spare parts which accompanies an aircraft squadron when it deploys to an advanced base.

applicable to requirements and distribution calculations.) We have frequently used a simple hypothetical example to illustrate our proposed method for designing these kits.[3] In this example we design a kit consisting of four different kinds of items, taking account of the demand probabilities and unit weights of the individual parts and of the total weight limit imposed on the kit. In the example the four items were assumed to be equally essential, and criticality was not considered.

Table 1 shows this same hypothetical example, except that the element of essentiality has been added. The four hypothetical items (A, B, C, and D) are indicated at the left. Items A and B have average demand rates of one per month, and items C and D have average demand rates of one third part per month or one part in three months. The unit weights of the items are 0.5, 5.0, 2.0, and 0.1 pounds. The essentiality factors assumed in the example are shown at the right. Item B is considered to be absolutely essential and thus has an essentiality factor of one. Item C is considered to be less essential and has an essentiality factor of one half. Items A and D are still less essential and are given essentiality factors of one fourth. Let us postpone, for the moment, the discussion of how one would go about obtaining essentiality estimates of this kind and instead look at one of the ways in which these estimates can be used if they are available.

TABLE 1
Hypothetical Flyaway Kit Allowing for Essentiality
(4-Line Items Only)

Line Item	Average 30-Day Demand	Unit Weight	Essentiality Factor
A	1.00	0.5	0.25
B	1.00	5.0	1.00
C	0.33	2.0	0.50
D	0.33	0.1	0.25

In designing a flyaway kit we have the task of choosing that combination of spare parts which will furnish the most supply support, but in doing so we must keep the total weight of our kit within a specified limit. In other words, as before, we want to minimize the expected number of shortages. Now, however, we will also weight each shortage by its seriousness.

The first step in this computation is illustrated in Table 2. The first column shows the unit of the particular item being considered. In other words, it indicates the first, second, third, and so on, units of the particular items. The next four columns indicate the probability that each unit of the four items will be used if it is available. The probability that the first unit of Item A will be needed is 0.632, or about 63 chances in 100. The probability that the second unit of A will be needed is about 26 chances in 100. The probability of needing the third unit of A is 8 chances in 100, and so on. The probabilities of needing each of the various

[3] See H. W. Karr and M. A. Geisler, "A Fruitful Application of Static Marginal Analysis," Management Science, Vol. 2, No. 4, July 1956; also M. A. Geisler and H. W. Karr, "The Design of Military Supply Tables," Operations Research, Vol. 4, No. 4, Aug. 1956.

TABLE 2

Computation of a Maximum Protection Kit

Unit Number	Probability of Use For Each Line Item				Additional Protection Per Pound Obtained From Each Line Item			
	A	B	C	D	A (0.5 Lb)	B (5.0 Lb)	C (2.0 Lb)	D (0.1 Lb)
1	0.632	0.632	0.283	0.283	1.264	0.126	0.142	2.827
2	0.264	0.264	0.045	0.045	0.528	0.053	0.023	0.451
3	0.080	0.080	0.005	0.005	0.160	0.016	0.003	0.050
4	0.019	0.019			0.038	0.004		0.004
5	0.004	0.004			0.008	0.001		
6	0.001	0.001			0.002			

units of Item B are the same as they are for Item A, because both items have the same demand rate. The probabilities for Items C and D are lower because they have lower demand rates. These probabilities are obtained for real spare parts from analysis of observed demand data, and the methods for doing so have been presented many times in the past, so here we can assume that the probabilities are given.[4]

These probabilities, then, indicate the additional protection (without regard to essentiality) of including each additional unit of each item in the flyaway kit. Imagine a situation in which flyaway kits are being designed for 100 different squadrons. In such a situation, if all of the squadrons were deployed at once, one would expect that about 63 of them would require at least one unit of Item A. Consequently, if the first unit of A were put into each kit, it would reduce the total expected number of shortages by 63. If the second unit of A were put into each of the 100 kits, it would reduce the total expected shortages by only 26. Consequently, the second unit of A is less than twice as valuable as is the first unit. The first unit of B is just as valuable as the first unit of A, since their demand rates are the same. The first unit of C, however, is less valuable than the first unit of A but more valuable than the second unit of A.

While the first unit of Item B provides as much additional protection as the first unit of Item A, it weighs ten times as much (5 pounds, as opposed to 1/2 pound). Consequently, it costs the kit more in terms of weight. To allow for the effect of weight, the additional protection obtained from each unit of each item is expressed on a per-pound basis. In other words, the probability that a unit will be needed is divided by its physical weight. This gives what we call the "additional protection per pound," and this is shown for each of the four items on the

[4]See B. B. Brown, Characteristics of Demand for Aircraft Spare Parts, The RAND Corporation, Report R-292, July 1956.

right-hand side of Table 2. The additional protection per pound for the first unit of Item A is 1.264, which is the probability that it will be needed (0.632) divided by the weight (0.5). Although the probability that the first unit of B will be needed is the same as for A, its additional protection per pound is only a tenth as great (0.126), because Item B weighs 10 times as much as Item A.

If all of the items were equally essential, we would now be in a position to select the items to go into our hypothetical flyaway kit. First, we would put in the part with the highest additional protection per pound (the first unit of D), then we would put in the part with the second highest additional protection per pound (the first unit of A), and so on until the total flyaway-kit weight limit had been reached.

But since our hypothetical items do differ in essentiality, it is necessary to adjust the additional protection value by multiplying them by their appropriate essentiality factors.[5] This computation is illustrated in Table 3. The left side shows the unadjusted values (from Table 2) of additional protection per pound. The right side shows these values after they have been adjusted by being multiplied by the essentiality factors. As you see, the adjusted protection values for Item B are unchanged, since its essentiality factor is 1.0; but the adjusted protection values are only half as large for Item C and only a fourth as large for Items A and D. At this point, the selection of the items to go into the kit can proceed as indicated earlier. The part with the highest adjusted additional protection per pound is included first, then the next highest, and so on, until the total weight limit has been reached. This procedure tends to

TABLE 3
Computation Adjusting for Essentiality

Unit Number	Additional Protection Per Pound Obtained From Each Line Item				Adjusted Additional Protection Per Pound For Each Line Item			
	A	B	C	D	A (E = .25)	B (E = 1.00)	C (E = .50)	D (E = .25)
1	1.264	0.126	0.142	2.827	0.316	0.126	0.071	0.707
2	0.528	0.053	0.023	0.451	0.132	0.053	0.012	0.113
3	0.160	0.016	0.003	0.050	0.040	0.016	0.002	0.013
4	0.038	0.004		0.004	0.010	0.004		0.001
5	0.008	0.001			0.002	0.001		
6	0.002							

[5]This procedure assumes that successive shortages of a particular item are of equal seriousness. This is a reasonable assumption in the case of flyaway kits, although it may not be in some inventory problems.

include relatively larger quantities of the items with high essentiality and relatively smaller quantities of the items with low essentiality. If our data are reasonably accurate, this procedure will result in a kit which gives maximum supply protection obtainable within the flyaway-kit weight limitation.

This example indicates one of the ways in which essentiality factors can be used in improving the quality of an important class of stocking-policy decisions. With minor modifications, the same type of factors are applicable to many other inventory problems. A similar procedure may even be applicable to problems of weapon-system design, since the system designer is also faced with the choice among many alternatives and capabilities subject to contraints of weight, size, and budget. Therefore, "essentiality" is an important factor to consider in making many kinds of decisions and should not be thought of only in connection with flyaway kits.

ELEMENTS OF ESSENTIALITY

Having seen one of the uses of essentiality factors, let us next consider the reasons why shortages of some items are more serious than shortages of other items. There are, in general, three main elements that determine the relative seriousness of a particular shortage: urgency, compensability, and mission effect.

Urgency - Some demands are less "urgent" than others for two reasons. First, as a matter of convenience, many items are replaced before the end of their useful life. For example, items that are routinely replaced every 500 hours can usually be left in for another 10 or 20 hours with little harm; a noisy radio may be usable for a little while longer. In such cases, the satisfaction of the demand can be postponed for a while. Consequently, the shortage penalty is less than if the part were completely inoperative. This particular effect might be taken account of by making adjustments in the demand data, but because it does not appear to be important we may ignore it and assume that all demands are urgent in the sense that the part has reached the end of its useful life. The other reason that makes some demands less urgent than others is that some items are simply not very important to the performance of the aircraft. This is really a part of what we call "mission effect," but I am including it here under "urgency" to simplify the remainder of the presentation.

Compensability - The second element of essentiality is the ability of the maintenance organization to compensate for the particular shortage in question. For this element we have coined the word "compensability." There are several ways in which shortages may be compensated for. These include such things as repair of the old item, local manufacture, use of substitutes, and cannibalization. A shortage of an item which is easy to compensate for is less serious than that of one difficult or impossible to compensate for.

Mission Effect - The third element which determines essentiality is the effect which the shortage has on the capability of the aircraft—in other words, the "mission effect." As we all know, some items would absolutely ground an aircraft if they were not available when needed, while others would merely limit its capability in varying degrees.

For purposes of illustration, let us consider only the last two of these three elements, namely, "compensability" and "mission effect," and not discuss "urgency" further (except as it refers to slight "mission effect"). Although both "compensability" and "mission effect" represent continuous variables ranging from very low to very high, let us arbitrarily classify all items into one of two categories of "compensability" and into three categories of "mission

effect." Figure 1 indicates such a classification containing five categories to which we have assigned arbitrary factors for purposes of illustration.

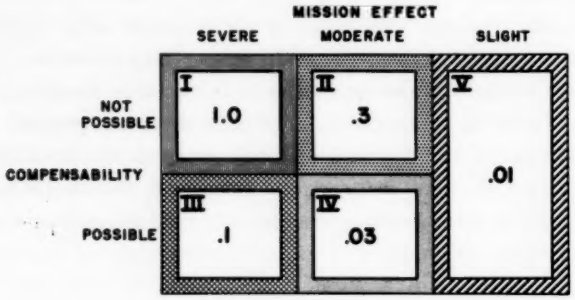

Figure 1 - Example of essentiality categories

The upper left-hand corner indicates a category of items with shortages impossible to compensate for and which severely affect the capability of an aircraft. Below it is indicated a category of items with severe "mission effect" but for which shortage compensation is possible. There are also two categories of items which only moderately limit the capability of the aircraft, one group of which can be compensated for and the other group of which cannot. A fifth category, made up of items which have slight effect on the capability of the aircraft, is indicated at the right. No distinction is made as to whether this last group of items can be compensated for, since they are not important enough to warrant the effort required in order to make the compensation.

The essentiality factors in each of the categories are merely illustrative and are not necessarily the ones which should be used in an actual computation. There are, in fact, two separate problems involved in making essentiality estimates. The first problem is to classify all of the various spare parts into however many categories it is decided to use, and the second problem is to assign an essentiality or cost factor to each of the categories.

The meaning of the five categories is perhaps more easily explained in terms of the schematic diagram of Figure 2. This diagram shows, in simplified form, the various actions and decisions which can result from a part shortage.

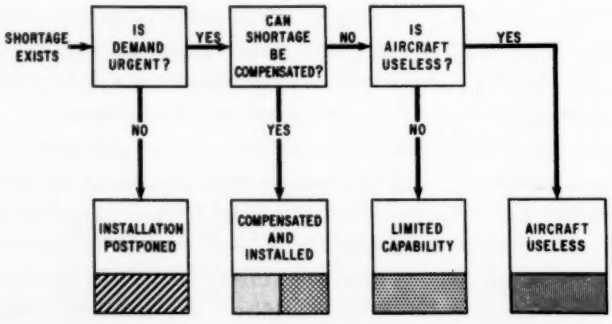

Figure 2 - Decisions resulting from a part shortage

Given that a shortage exists (upper left of diagram), the first question is whether or not the demand is urgent. If the item has slight "mission effect" the answer is "no," the installation of the item is postponed, and only a small shortage penalty is incurred. In Figure 1 we assigned a factor of 0.01 to this category.

If the demand is urgent, the next question is whether or not it is possible to compensate for the shortage. If compensation is possible by local manufacture, substitution, repair, or other means, then the compensation is made and the item installed. In this case the shortage penalty would be higher than in the previous case. In Figure 1 we assigned factors of 0.10 and 0.03, depending on the "mission effect" of the particular item.[6]

If the shortage cannot be compensated for, the next question is whether or not the aircraft is useless without the item. If it still has a limited capability, the shortage penalty would be lower than it would have been had the shortage made the aircraft useless, but it would be higher than it would have been had it been possible to compensate for the shortage. In Figure 1 we assigned factors of 0.30 to "noncompensable" items with moderate "mission effect."

This schematic diagram indicates how one can develop a set of categories of spare-part essentiality, and, as can be seen, the essentiality of the categories ranges from very low to very high.

THE INITIAL ESTIMATING PROCEDURE

Let us turn next to the problem of how, given a list of spare parts, one can proceed to sort them into proper categories of essentiality. As is indicated in the flow chart of Figure 2, the effect of the shortage will depend quite largely on decisions made by maintenance personnel. The problem then is to predict in advance what these decisions will be in order to adjust the composition of the parts inventory accordingly. It seems to us that the only feasible way of doing this is by human judgment.

Psychologists and sociologists have long been working on the problem of measuring and classifying human judgments. Consequently, we enlisted the aid of two experts in psychometric scaling and proceeded to apply one of the psychometric techniques to the logistics problem of estimating spare-parts essentiality. The remainder of our discussion will deal with an experiment in which such methods were used to classify F-86H spare parts into categories of essentiality.[7]

The following is a list of the various things which are needed to make a judgmental classification of essentiality:

1. List of spare parts to be rated
2. Group of judges
3. Pictures identifying parts
4. Essentiality questionnaire
5. Scaling technique.

First of all is needed a list of the parts which are to be rated. In the experiment at Clovis Air Force Base, we selected a sample of 417 items from the T. O. 1F-86H-4 and related handbooks. We grouped these items under eight different specialties, such as "general aircraft," "engine," "hydraulics," etc., so that we could use specialists as judges.

[6] The reason that "mission effect" affects the essentiality of a "compensable" item is that there is some risk that the compensation might not be made because of lack of time, etc.; thus the condition of the aircraft before the compensation is accomplished is of importance.

[7] This experiment was conducted at Clovis Air Force Base by two members of the RAND Psychological Research Department, David G. Hays and William W. Haythorn.

The judges used in this particular experiment consisted of all the senior mechanics and specialists at Clovis Air Force Base who had had adequate experience to enable them to give reliable answers. (The manufacturers' maintenance engineers might be used in rating parts for brand new aircraft.) For example, there were 13 judges for the "general aircraft" items. Twelve of them were master sergeants or technical sergeants. The thirteenth was a contractor's technical representative. All of the sergeants had had combat theater experience and had over a year's experience on the F-86H. The other specialities—engine, hydraulics, etc.—were less well represented, although adequately so for this particular feasibility test.[8]

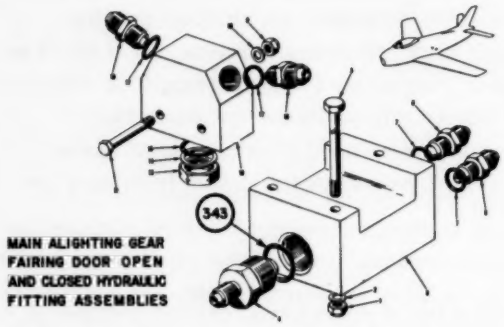

Figure 3 - Form of presentation for estimating procedure

The pictures used to identify each item were illustrated drawings taken from the same handbooks as were used to draw the sample and were presented to the mechanics in the way illustrated in Figure 3. Each specialist was given a book of these drawings. The pictures permitted him to see what the items looked like as well as where they went in the aircraft. Short verbal descriptions also were attached. Only one item on each page was to be rated, and it was indicated by a large circled number which corresponded to the number on an IBM punch card. This numbered card was sorted into one of several pockets by the reviewing specialist to record his answer to the questionnaire.

In the Clovis experiment we used three different questionnaires, each of which was designed to measure one of the major elements of essentiality ("urgency," which I am omitting from this discussion, "compensability," and "mission effect"). Each specialist answered each of these questionnaires for every sample item in his specialty.

The "compensability" questionnaire read: "Suppose that your squadron has been transferred to a newly activated, advanced tactical base. Your aircraft are flying regular combat missions. After one such mission, a part is inoperative. Your supply unit does not have a replacement in stock. The aircraft is needed for combat the next day. Can you get it into combat readiness?" Each specialist was then asked to indicate one of the following answers for each part in his book of pictures. He did so by dropping the numbered IBM cards into an appropriately marked pocket. The possible answers were:

(A) No, the item is not installable.
(B) Yes, but only if I can cannibalize.
(C) Yes, but it depends on having special tools available.
(D) Yes, I can repair that part or make up a new one with material I'm sure to have.
(E) Yes, the aircraft is combat-ready with that part inoperative.

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[8]During the pre-test phase of this experiment, the desirability of using aircraft pilots as judges was investigated. But the pilots felt they did not have the detailed knowledge of spare parts required to make the judgments and that maintenance specialists could give better estimates.

The questionnaire for "mission effect" set somewhat the same scene, except that it excluded the possibility of compensating. It read: "Suppose that your squadron has been transferred to a newly activated, advanced tactical base. Your aircraft are flying regular combat missions. After one such mission, a part is inoperative. Your supply unit does not have a replacement in stock, and you are not able to make the part operate again. What is the status of this aircraft?" The possible responses were:

(A) The aircraft cannot fly; it would have to be destroyed if the base were evacuated.

(B) The aircraft can fly, but only in an emergency (such as evacuation).

(C) The aircraft can fly, but using it involves a calculated risk. It should be used only in a critical operational situation.

(D) The aircraft can fly well enough to go on routine combat missions, but its capabilities are somewhat limited.

(E) The inoperative part has practically no effect on the operation of the aircraft.

As in the case of the "compensability" questionnaire, each specialist went through his entire book of illustrations, selecting one of the five answers for each part indicated in his book. He recorded his choices by dropping a numbered IBM card into one of five pockets of a card rack. In this way, we were able to collect from several different specialists independent essentiality estimates for each part in our sample. At this point we applied one of the psychometric scaling techniques in order to combine these individual judgments into a single set of estimates which we feel are superior to what any one man could have done by himself.

Rather than go into detail about the technique of scaling, we have a simplified hypothetical example which will at least give some of the flavor of the technique used in this particular study. Table 4 shows the results of a fictitious case where there are only three judges and five parts to be rated. (Actually, more judges and more parts are required to give meaningful results.) Let us assume that these three men have rated the five parts in response to the "mission effect" questionnaire. Although these data are fictitious as a matter of convenience, they represent a typical situation.

TABLE 4
Hypothetical Example of Scaling Procedure

Parts	Classification		
	By Judge 1	By Judge 2	By Judge 3
3	A	A	B
5	B	C	B
1	C	B	B
2	D	C	D
4	E	E	E

The first thing we notice in this example is that the three hypothetical judges do not agree in their classifications. This will almost always be the case, especially if there are many parts and many judges. But even though the judges do not agree exactly on the classifications, they do tend to rank the items in roughly the same order, although even these rankings do not agree exactly. For example, judge number 2 has reversed the order of parts 1 and 5 in relation to the way they are ordered by judge number 1, while judge number 3 feels that parts 1, 3, and 5 are equal.

In order to obtain a final set of categories and classifications we look for specific divisions or discriminations on which there is agreement. The most obvious one in this case is part number 4. All three judges agree that it has a less severe "mission effect" than any

of the other parts, so we can draw a line separating that part from the others and say that it falls into our lowest "mission-effect" category.

The case of part number 2 is less clear. Two of the judges call it "D" while one of them calls it "C". But since judges 1 and 3 agree that it has less severe "mission effect" than do parts 1, 3, and 5, and since judge 2 says that its "mission effect" is no more severe than that of parts 1, 3, and 5, we follow the majority rule (which is easier to justify when we have more judges) and draw a line separating part number 2 from the others. We will say that part number 2 falls into our second lowest category of "mission effect."

The discrimination between parts 1 and 5 appears to be completely ambiguous. Judges 1 and 2 disagree as to which of the two parts (1 or 5) has more severe effect; and judge 3 thinks that they have equal effect. Consequently, we make no distinction between parts 1 and 5. There is majority agreement, however, that both parts have less "mission effect" than part 3, so we can draw a line separating part 3 from the other parts and thereby distinguish four categories of mission effect. (Part 3 is in the first category, parts 1 and 5 are in the second, part 2 is in the third, and part 4 is in the fourth.)

There is actually quite a bit more to the process of combining the individual judgments than I have just indicated. The problem is more complicated with more parts and more judges, and there are additional refinements which will improve the accuracy of the scale beyond that achievable by the simple procedure I have just described. This example, however, illustrates the general idea. If more detailed information about the specific techniques is desired, there is available literature written by experts on psychometric scaling.[9] The main point to be drawn from this hypothetical example is that it seems better to use the group judgment of three capable, experienced judges than to use the judgment of any one of them alone.

An alternative method of combining individual judgments, namely, the conference technique, also appears to be very satisfactory, provided it can be based on a set of categories and definitions which have been obtained through a detailed study such as the Clovis experiment. The conference has the advantage of being easier to organize and administer and it can make the classifications in less time than is required by the method of collecting and combining individual judgments. The conference is much more difficult to analyze, however, and is therefore not as suitable for the initial investigation which is required to identify the relevant variables and to establish a set of essentiality categories.

This combined approach of a preliminary detailed study to establish categories and definitions, followed by a conference to classify the entire list of items, was followed by the Sacramento Air Materiel Area in their F-86H and F-100D flyaway-kit project.

RESULTS OF THE CLOVIS EXPERIMENT

The preceding discussion indicates the way in which the Clovis essentiality data were collected and analyzed. Let us now consider some of the results of this analysis.

In Figure 1 we presented a set of five essentiality categories resulting from two levels of "compensability" and three levels of "mission effect." Each of the spare parts in our sample was classified into one of these five categories. The following illustrations show a few examples and their classifications.

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[9]See S. A. Stouffer, L. Guttman, E. A. Suchman, P. F. Lazarsfeld, S. A. Starr, and J. A. Clausen, The American Soldier, Vol. IV, Measurement and Prediction, Princeton University Press (1950); also S. A. Stouffer, E. F. Boratta, D. G. Hays, and A. F. Henry, "A Technique for Improving Cumulative Scales, "Public Opinion Quarterly, Vol. 16, No. 2, 1952.

The first example is a nose-wheel casing (Figure 4). Of the 13 general-aircraft mechanics who rated this item, 11 answered "B" on the "compensability" questionnaire ("Yes, but only if I can cannibalize"), one answered "A" ("No, the item is not installable"), and one didn't answer. On the "mission effect" questionnaire, eight of the judges said the aircraft couldn't fly (answer "A"), two of them said it could be evacuated (answer "B"), and three of them didn't answer. There was exceptionally good agreement that this item was either an "A" or a "B" on both scales, so we classified it as being "noncompensable" with "severe mission effect."[10] This item should clearly be assigned a very high essentiality factor, and we have assigned a value of 1.0 or 100 percent to items in this category.

The second example is a fuse-assembly cover (Figure 5). The 13 mechanics agreed equally well that this item belonged under answer "E" in both questionnaires. Consequently, we classified it as having very little effect on the mission capability of the aircraft. To items of this type, we have assigned an essentiality factor of 0.01, which is 1 percent as much as we have assigned to "noncompensable, severe-mission-effect" items. These first two examples are rather obvious ones.

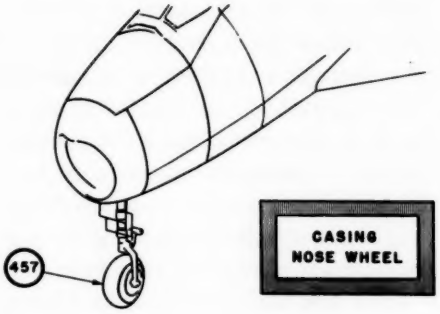

Figure 4 - Example of noncompensable item with severe mission effect

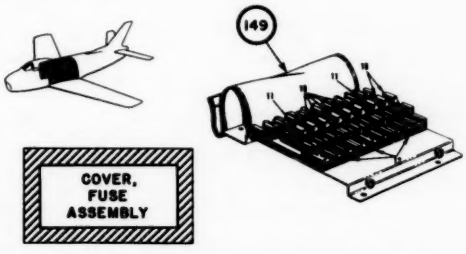

Figure 5 - Example of item with little mission effect

The classification of some of the items was not so obvious. For example, the gasket which we saw earlier in explaining how the parts were identified (Figure 3) was classified as having "severe mission effect" but as being easily "compensable." Another item (Figure 6), an air compressor, was classified as having only "slight mission effect." These last classifications might seem unlikely to people not familiar with the

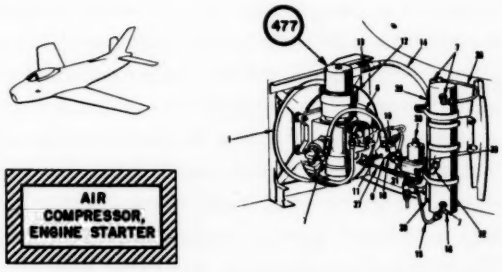

Figure 6 - Example of item with little mission effect

[10]Items were classified "noncompensable" if they received answers "A", "B", or "C" on the compensability questionnaire, and they were classified "compensable" if they received answer "D". Answers "A", "B", and "C" on the mission-effect questionnaire were taken to indicate "severe mission effect," and answer "D" was taken to indicate "moderate mission effect." Answer "E" on either questionnaire classified the item in the "slight-mission-effect" category.

PROPERTY CLASS	ESSENTIALITY CATEGORY I	II	III	IV	V
01	26	15	6	16	37
03	14	34	3	12	37
04 & 29	11	0	18	37	34 ♦
11	0	0	0	50	50
16	46	19	4	25	7
ALL CLASSES	21	17	6	24	32

Figure 7 - Distribution of essentiality by property class (percentage of sample items in each class)

UNIT COST (DOLLARS)	ESSENTIALITY CATEGORY I	II	III	IV	V
.01	1	6	7	8	♦29
.10	8	6	3	♦17	21
1.00	10	11	5	♦20	24
10.00	19	♦11	1	7	10
100.00	11	♦6	1	1	4
1000.00	♦4	0	0	0	2

Figure 8 - Relationship between essentiality and unit cost (number of items in F-86H sample)

details of the aircraft, yet there was very good agreement among these mechanics as to their "compensability" and their "mission effect."

Figure 7 shows the way in which essentiality was distributed in our Clovis sample within a few selected property classes (01, airframe; 03, accessories; 04-A and 29, hardware; 11, armament; and 16, communication) and also the way in which it was distributed over all classes combined. The essentiality categories from high to low are indicated across the top and the property classes down the side. Looking first at the distribution for all classes combined (the bottom row of the figure), we notice that there is a sizeable percentage of the items in each of the essentiality categories. Assertions that all items are equally essential do not appear to be borne out by these data, and the differences in essentiality appear to be important enough to warrant consideration.

It is also interesting that there appear to be some differences among property classing in terms of essentiality. While the airframe, accessories, and communications parts seem to be spread across the board, there is a noticeable bunching of the hardware and armament parts at the low-essentiality levels. In the case of hardware parts, this seems reasonable, since they might tend to be easy to compensate for and have low "mission effect." The concentration of the armament parts in the low-essentiality categories is partially due to the fact that our sample was drawn from the T. O. 1F-86H-4 handbook, which contains only the attaching hardware for the armament components. It also is probably true that a large proportion of the items in the class consists of clips, brackets, washers, etc., which are the sorts of items which occurred in this particular sample. On the basis of the data we already have, it seems likely that each of the relevant property classes contains items with very high essentiality and items with very low essentiality.

Figure 8 shows the relationships between essentiality and unit cost for the items in our sample for which we could find prices. The first thing to notice in these data is that there is a noticeable direct relationship between unit cost and essentiality. (The same relationship is also observed between unit weight and essentiality, since weight and cost are positively correlated.) To emphasize this relationship, we have indicated with an arrow the essentiality category which contains the median or middle item of each unit cost group. It is clear that there is a strong tendency for inexpensive or lightweight items to have low essentiality and for expensive or high-weight items to have high essentiality. This is, of course, to be expected, and we would have been surprised if it had turned out otherwise.

An interesting thing about this relationship between essentiality and unit cost or weight is that if all of the items followed this tendency quite closely there might be very little to gain from taking account of unit cost or unit weight in our inventory decisions, because the effect of essentiality and the effect of cost would tend to cancel each other. But the data in Figure 8 indicate that while the average high-cost item has high essentiality and the average low-cost item has low essentiality, there are very many exceptions which should be taken account of.

In this sample there are 20 or so items with very high essentiality and very low unit cost. These items, which include such things as capacitors, coils, and resistors, should be stocked in relatively larger quantities than other low-cost items with lower essentiality.

On the other hand, there are several high-cost items which have low essentiality. Two items costing over $1000 are such that the mechanics think they could get along without them. One of these items is the compressor, shown earlier in Figure 6 as an example of an item with very low essentiality. The other is a gun camera.[11] It is the existence of this large number of exceptions which makes it worthwhile to take specific account of essentiality.

ASSIGNING NUMERICAL VALUES TO ESSENTIALITY CATEGORIES

As already mentioned, the somewhat arbitrary numerical essentiality factors in the preceding example were assigned for illustrative purposes. But these factors can be measured quantitatively in terms of dollar cost if it is desired to go to the additional effort of conducting a detailed cost analysis. If we have an estimate of the cost of each class of events, it is a simple matter to convert them to the kinds of factors we have indicated in Figure 1 or to leave them in dollar-cost form for use as shortage penalties in the various provisioning and distribution formulas in which they are needed. In fact, the way that we arrived at the essentiality factors in the previous illustration was first to guess at the dollar cost of each kind of shortage, for it is easier to think first in terms of cost and then to convert the relative costs to decimal fraction.

The costs referred to do not, of course, represent negative values of "military worth," because it is not possible to measure "military worth" in dollar terms. It is frequently possible, however, to estimate the dollar cost of accomplishing a particular military objective, even though the value of the objective cannot be expressed in dollars. Consequently, it is also possible to measure the cost of accomplishing the objective when various kinds of shortages occur. Suppose, for example, the objective is to maintain a particular readiness posture in terms of mission-ready aircraft as well as to maintain a given training program. In this case it is at least conceptually possible to estimate the additional cost required to maintain this objective when shortages in each essentiality category occur. For example, consider the case of the highest essentiality category we have defined ("noncompensable, severe mission effect"). If a shortage of this kind occurs, it would require the availability of an additional aircraft for the duration of the shortage if the objective is to remain unchanged. Consequently, the dollar cost of this kind of shortage is equal to the additional or marginal cost of the extra aircraft availability. In the case of a fighter-bomber, we guessed that this cost is at least $1000 a day.

[11] The fact that these items are classified into the lowest-essentiality category does not mean that they are "not essential"; it merely means that they are "less essential" than the other items. The air compressor mentioned above is undoubtedly well worth its cost, but it is not as important as a wheel casing, since even at the most austere advanced base the squadron would have to have a ground air compressor which could also be used to recharge the air bottle. Similarly, the gun camera is certainly useful, but it is not as essential as a piece of the landing gear. Consequently, the less-essential items should be stocked in relatively smaller quantities than the items with high essentiality.

On the other hand, if the shortage had "severe mission effect" but could be compensated for (our essentiality category III), the cost of maintaining the readiness and training objective would be considerably less. The main additional cost would be the cost of making the compensation (additional manhours, equipment, material, etc.) plus some need for extra aircraft availability to offset any delays which may occur. We guessed that this might be of the order of $100. In the lowest essentiality category, we assumed a small inconvenience cost of $10, and we assumed that shortages having "moderate mission effect" would require $300 or $30 additional cost to restore the objective to its stated level, depending on the compensability of the item.

If an actual analysis were made to estimate the last two costs, it would be necessary to determine the exchange ratio between a fully mission-ready aircraft and an aircraft with limited capability. This exchange ratio would have to be based on the preferences of the commander. But since only reasonable approximations are needed for most inventory control purposes, such an estimate would be adequate. In fact, it is my opinion that it is probably not worth the effort involved to make a detailed cost analysis at all; instead, it is better to rely on educated guesses as to the relative shortage penalties involved in each item class of events described by the essentiality categories. Once the items have been sorted out from high to low essentiality and a reasonable set of penalty factors have been attached, most of the benefit of considering essentiality will have been achieved. Additional refinements are probably not worth the cost involved.

CONCLUDING REMARKS

The preceding discussion has indicated the main features of our proposed procedure for classifying spare parts according to their essentiality or shortage penalties. There are many ways in which these estimates can be used to improve the effectiveness of a spare-parts inventory. Essentiality estimates can also be used to evaluate supply management performance.

One of the main results of this experiment was that it produced a set of essentiality categories with illustrated examples which appear to be applicable to all tactical fighter-bombers. This makes it possible to use a less-expensive conference procedure in classifying a full range of spares. In other words, it is only necessary to conduct one detailed investigation of a sample of spare parts for a particular type of aircraft and mission. Thereafter, a conference can make the classifications on the basis of the categories and definitions obtained from the detailed study. For example, a conference at Sacramento Air Materiel Area, in which the F-86H results were used as a model, recently completed the essentiality classification of the full range of F-100D parts.

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AIRLINE DEMAND:
AN ANALYSIS OF SOME FREQUENCY DISTRIBUTIONS*

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Following some trial fittings of Poisson and compounded Poisson distributions, Gamma distributions are fitted to histograms of manifested passengers, to total losses (late cancellations, no-shows, and misconnections) and to passengers added.

INTRODUCTION

The scheduling of departure times for flights on a given route is a complex problem involving many considerations. One of the simplest questions concerns the optimal over-all capacity of flights on a route for a given period. Obviously this problem requires a balancing of the revenue obtainable with a certain capacity against its cost. Now passenger demand is a random variable. The mean value of revenue as a function of capacity has the shape of a saturation curve, related in a simple manner to the distribution function of passenger demand: it is the expected value truncated at the capacity level. For a more sophisticated calculation of expected revenues, the probabilities of cancellations, of no-shows and misconnections, and of stand-by demand should be considered separately. It is thus of some value to know the actual passenger frequency distributions on typical flights. In this paper some empirical frequency distributions will be presented, together with some theoretical distribution functions that have been fitted to them.

Some obvious difficulties should be pointed out first. The "permanent factors" affecting demand rarely remain constant over significant periods, changes of schedule and of equipment occurring every few months. In the past, total demand for airline transportation has also been subject to a substantial upward trend. On flights with a heavy demand relative to capacity—the most interesting ones as regards the decision problem—only part of the demand distribution is observable, the frequency of demands for more seats than capacity appearing in the aggregate only.

The data used were the daily load reports of one airline covering the period from September 1954 through May 1955, listing for each flight the number of passengers manifested (i.e., on the passenger list a few hours before departure). Available for shorter periods were

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We are indebted to the airline which kindly supplied us with the data for this analysis, and particularly to Messrs. McQuillen and Lets from whose advice we have greatly benefited.

†Manuscript received April 10, 1957.

data on the various losses (late cancellations, no-shows, and misconnections) and, for a few months, the passengers added.

1. THE DEMAND FOR RESERVATIONS

1.1 The Poisson Approximation

Flying to a particular destination on a particular day may be regarded a "rare event" for any given passenger. The probability distribution of the total demand for a flight may therefore be assumed related to the Poisson distribution for rare events. More precisely we may argue as follows: let the time period during which reservations may be made for a flight be divided into a number of intervals such that the mean number of requests for reservations received during any one interval is the same. Let time now be measured in terms of these interval units, one interval being one time unit, and consider the probability of receiving a call for a reservation during a small time interval. If the population of potential customers is large (consequently, if the probability of any one desiring to go on this flight is small) then the number of reservations already received does not affect the probability of further requests for a reservation. Suppose, to begin with, that reservation requests are made for one seat at a time. If the interval considered is sufficiently small, the probability of getting one reservation request is proportional to the length of the interval, and the probability of getting more than one request is zero. It is well known [Ref. 1, pp. 115-118] that on these assumptions the accumulated reservations over any fixed time interval of finite length must then be Poisson distributed. Let n be the number of reservations. Then the probability f(n) of n reservations is

$$f(n) = \frac{\lambda^n e^{-\lambda}}{n!}, \tag{1}$$

where $\lambda > 0$ is a constant, the mean of n. In particular, this must be true for the total reservation requests that have accumulated at flight-departure time if we count as requests only those that are not cancelled later.

Examination of data restricted to seasons and to such weekdays that have an approximately stationary demand shows that there is some tendency for the empirical distributions to approximate the Poisson distribution, but the fit is rather poor (Figures 1 and 2). The usefulness of this approximation is also limited because one would not wish to distinguish too many different cases of weekdays and seasons as would be required in order to arrive at a "pure" situation.

1.2 Poisson Process with Clustered Demand

Histograms of passengers manifested seem to show that on certain flights demand occurs more frequently for an even number of seats than for an odd number (Figure 2). This fact may be explained by assuming that requests for reservations obey a Poisson distribution, as before, but that each request is for either one or two seats, with constant probabilities. If we assume that p is the probability that one seat is requested and $q = 1 - p$ the probability of a request for two seats, the probability of a total demand for n reservations can be shown to be

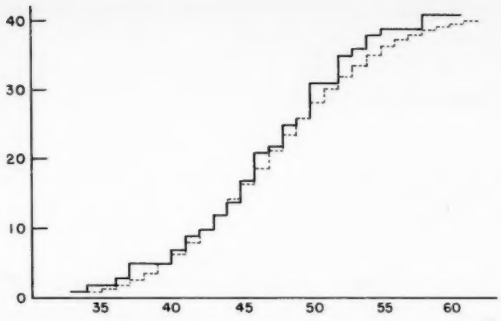

Figure 1 - Flight X, September 7 - November 18, 1954, Tuesday through Friday only.
Dotted line: Poisson cumulative, $\lambda = 46.3$.
Solid line: Observed cumulative.

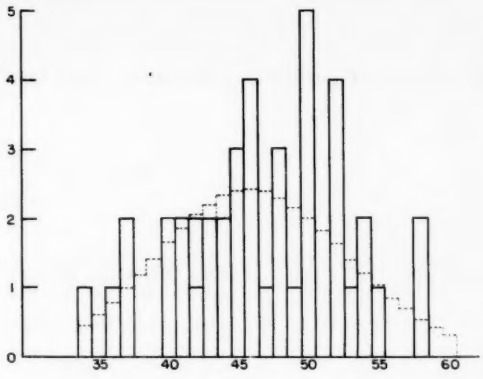

Figure 2 - Flight X, September 7 - November 18, 1954, Tuesday through Friday only.
Dotted line: Poisson frequency distribution, $\lambda = 46.3$.
Solid line: Observed frequencies.

$$(2) \qquad p(n) = e^{-\lambda} \sum_{k=n-[n/2]}^{n} \frac{\lambda^k q^{n-k} p^{2k-n}}{(n-k)!\,(2k-n)!}.$$

where $[a]$ denotes the largest integer not greater than a. Therefore, the smallest integer greater than or equal to $[n/2]$ is $n - [n/2]$. The number of terms in the expression for $p(n)$ increases by one when n increases by one from an odd to an even number and does not change when n increases from an even to the next odd number. This indicates a tendency for even-numbered demands to have a larger probability.

Formula (2) is cumbersome. A more symmetric expression can be obtained on the assumption that demand occurs also for more than two seats at a time and that the clustering is subject to some simple probability distribution. Among plausible candidates, the geometric distribution is perhaps the simplest one. In order to achieve a rapid decrease of the probability of larger clusters, a rather considerable probability would be required for demands of one. It is natural now to associate the probabilities of the ordinary geometric distribution with demands that are one unit larger. Then the probabilities $p(k)$ of a demand for k seats are given by

$$(3) \qquad \begin{aligned} p(0) &= 0 \\ p(k) &= (1 - p)\, p^{k-1} \qquad k > 0\,, \end{aligned}$$

where p is a constant, $0 < p < 1$.

The probability of a total demand for n seats may now be derived as follows. The generating function of the Poisson distribution is $e^{-\lambda+\lambda s}$, that of the ordinary geometric distribution is $\frac{1-p}{1-ps}$, and that of the geometric distribution as used here is $\frac{(1-p)s}{1-ps}$. We are interested in sums of geometrically distributed variables for which the number of terms obeys a Poisson distribution. The generating function of the distribution of these sums is obtained by substituting in $e^{-\lambda+\lambda s}$ the expression $\frac{(1-p)s}{1-ps}$ for s [Ref. 1, p. 223]. Thus

$$\Phi(s) = e^{-\lambda} + \lambda \frac{(1-p)s}{1-ps}$$

is the desired generating function. This may be expanded as follows.

$$\Phi(s) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k s^k}{k!} \left(\frac{1-ps}{1-p}\right)^{-k}.$$

Developing $(1 - ps)^{-k}$ into a binomial series, we have

$$(1 - ps)^{-k} = \sum_{i=0}^{\infty} \binom{-k}{i} (-ps)^i .$$

Now
$$\binom{-k}{i} = (-1)^i \frac{k\,(k+1) \ldots (k+i-1)}{i!}$$

$$= (-1)^i \binom{k+i-1}{i}.$$

Thus
$$(1 - ps)^{-k} = \sum_{i=0}^{\infty} \binom{i+k-1}{i} (-p)^i s^i ,$$

and, finally,

$$\Phi(s) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{i=0}^{\infty} \binom{i+k-1}{i} p^i s^{i+k}.$$

Now let $i + k = n$

$$\Phi(s) = \sum_{n=0}^{\infty} \left[e^{-\lambda} \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} \lambda^k p^{n-k} \right] s^n .$$

This shows the probability of a demand n to be

(4)
$$f(n) = e^{-\lambda} \sum_{k=1}^{n} \binom{n-1}{n-k} \frac{1}{k!} \lambda^k p^{n-k}.$$

The mean and variance of this distribution are

$$\mu_1 = \frac{\lambda}{1-p},$$

$$\mu_2 = \frac{\lambda(1+p)}{(1-p)^2}.$$

These two moments of the distribution were fitted to the histogram of one flight* with a sufficiently low load factor in order that all demands should fall within the capacity limits of the flights, and a maximum number of observations became available (Flight Y, Figure 3). A chi-square test did not reject the distribution at the 5-percent significance level. But an unexpectedly large value of p was obtained, namely p = .50, so that it remains doubtful that the distribution (4) really explains the observed frequencies. This p-value would predict probabilities $\left(\frac{1}{2}\right)^i$ for a demand cluster of size i.

1.3 The Gamma Distribution

The observed frequency distributions of manifested reservations tend to be more dispersed than Poisson distributions. This condition is probably due not only to demand clustering but also to variations in the mean demand that are caused by random events such as weather, holidays, or conventions. This would mean that the Poisson parameter λ should itself be regarded a random variable. If, for instance, λ obeys a Gamma distribution, then it is well known [Ref. 2, p. 125] that the variable n follows a negative binomial distribution

$$f(n) = \binom{-r}{n} (1-p)^r (-p)^n, \tag{5}$$

where p is a constant, $0 < p < 1$, and r a positive integer.

Compounding this distribution for the number of reservation requests, with different distributions for the number of seats per request, we can obtain various distributions for the total number of reservations but involving rather complex expressions. Rather than introducing the Γ-distribution in this way by the back door, so to speak, we shall use it for a direct approach to the distribution of total reservations.

For the cumulative distributions we use the notation

$$\int_0^x \frac{\alpha^k u^{k-1}}{\Gamma(k)} e^{-\alpha u}\, du = \int_0^{\alpha x} \frac{u^{k-1} e^{-u}}{\Gamma(k)}\, du = \Gamma_{\alpha x}(k). \tag{6}$$

The mean and variance of the Γ-distribution are, respectively,

$$\mu_1 = \frac{k}{\alpha}, \qquad \mu_2 = \frac{k}{\alpha^2}.$$

*A change of schedule did not permit continued observation of flight X.

The mean of the hypothetical Γ-distribution was fitted to the observed means and then a maximum likelihood estimate of the exponent k was obtained for two flights (Figures 3 and 4).

In view of the fact that no observations were left out for the entire 10-month period, which included the winter season, the agreement is good enough to accept the Γ-distribution. More work would be needed to establish its universal applicability.

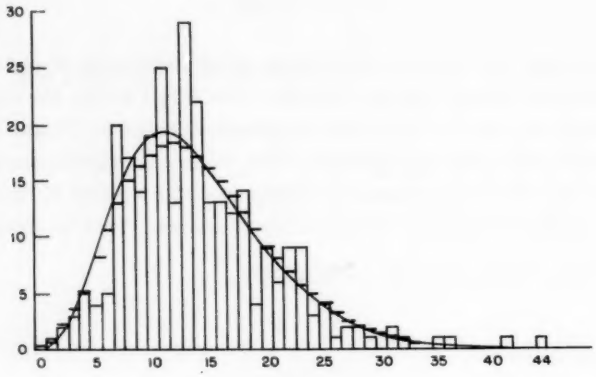

Figure 3 - Flight-Y manifests.
Curved line: Maximum likelihood fit of $(x + 1/2)$ to Γ-density; $\frac{\alpha^k}{\Gamma(k)} x^{k-1} e^{-\alpha x}$; $\hat{k} = 4.94991$, $\hat{\alpha} = 0.148541$.
Heavy short line: Method of moments fit to compound Poisson with geometric. Parameter in geometric, 0.506; parameter in Poisson, 6.903.

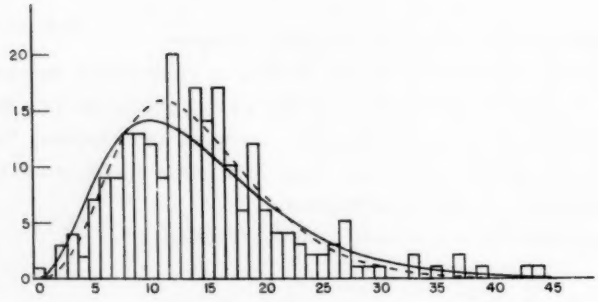

Figure 4 - Flight-Z manifests; Γ-distribution.
Dotted curve: Maximum likelihood estimate; $-\hat{\alpha} = 0.34$, $\hat{k} = 4.96$.
Solid curve: Moments estimate; $-\hat{\alpha} = 0.25$, $k = 3.69$.

2. LOSSES

Histograms of late cancellations, of misconnections, and of no-shows are presented without comment (Figure 5). The number of days for which data were available in useful form was limited and not sufficient to permit conclusions about the nature of the distributions involved.

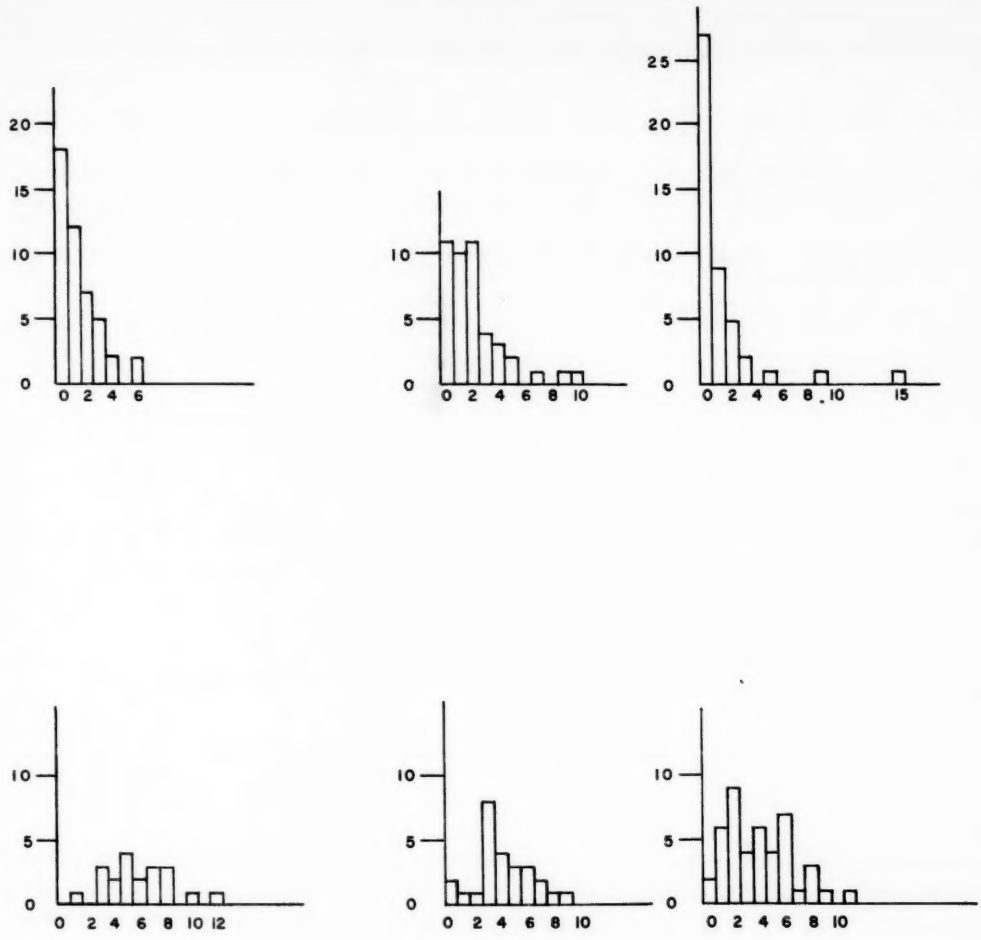

Figure 5 - Histograms of late cancellations, misconnections, and no-shows

Since late cancellations and misconnections are usually small compared with no-shows, the distribution of total losses may be regarded as a modified distribution of no-shows. Since it is the total losses that matter in the decisions mentioned later (the sale goal and the stop-sales problem) their distribution has been studied more carefully (Figures 6 and 7). The approximation by Γ-distributions seems satisfactory, and the exponents turn out close to each other and to those of the demand distributions.

3. DEMAND OF STAND-BY PASSENGERS

The population of stand-by passengers (adds) consists of through passengers who missed connections or changed plans and of local passengers who either did not bother about making a reservation or took a chance upon being unable to secure a reservation in time. The superposition of these demand elements may produce complex-shaped distributions. The two histograms (Figure 8) show unimodal distributions to which Γ-distributions have been fitted.

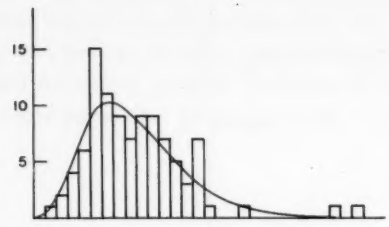

Figure 6 - Flight-Q total losses for manifests of 50 and over. Γ-distribution using $(x - 1/2)$; $\hat{\alpha} = 0.478$, $\hat{k} = 4.257$.

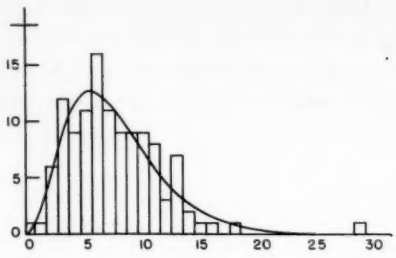

Figure 7 - Flight-P total losses for manifests of 50 and over. Γ-distribution; $\hat{\alpha} = 0.469$, $\hat{k} = 3.733$.

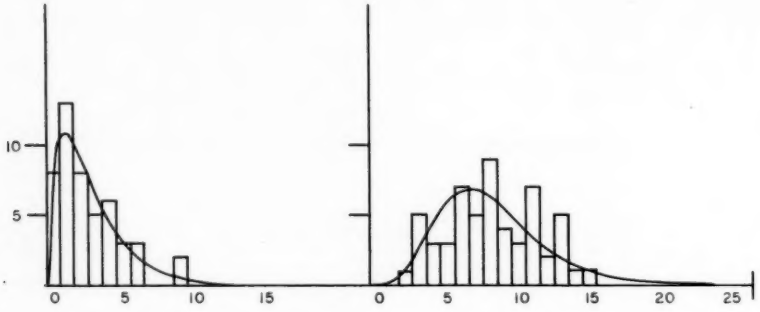

Figure 8 - Histograms of passengers added to which Γ distributions have been added

The observations cover a period of only two months and of only such days for which demand was low enough to permit the total demand of stand-bys to be accommodated. Unfortunately, this excludes from observation the demand of passengers "unable to secure a reservation"; for this demand would show up mainly on days when not all available adds can be accommodated.

CONCLUSIONS

Some further possibilities of applying demand distributions may be pointed out in conclusion. In choosing between types of equipment one must compare costs and expected revenues. As we have seen, the latter are proportional to a truncated expected value dependent on the demand distribution.

In setting an optimal price of a noncompetitive flight, the demand distribution should be considered for its effect on both revenue and on the passenger's average chance of obtaining a reservation. This has been discussed elsewhere [3]. Finally, some delicate questions of reservations' policy cannot be approached without information as to the distributions of reservations' demand, late cancellations, no-shows, and stand-by demand. These questions which revolve around the problem of keeping oversales below a certain limit, while simultaneously reducing lost revenue from no-shows to a minimum, have been investigated at length in another paper [4].

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[4] M. Beckmann, "Decision and Team Problems in Airline Reservations," Econometrica Vol. 26, No. 1 (Jan 1958)

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DYNAMIC PROBLEMS IN THE THEORY OF THE FIRM*

H. M. Wagner and T. M. Whitin

This article was published as Appendix 6 in the second edition (1957) of the book entitled "The Theory of Inventory Management," by Thomson M. Whitin, in order to reach an audience which might otherwise not be reached in the field of logistics. The Princeton University Press, who publish this book and who are the copyright owners, have granted permission to reprint the above material in the Naval Research Logistics Quarterly.

There is a current widespread interest among economists in the solution of dynamic problems in the theory of the firm.[1] As soon as we relax the usual assumption that the firm faces a steady state rate of demand and schedule of costs, we are forced to consider patterns of production and sales which vary over time and which allow for the possibility of inventory holdings. We find in the recent and numerous contributions to the dynamic theory of the firm a concentration on two cost variables in addition to the standard period costs of production, viz., the cost of holding inventories and the cost of changing the rate of production between time periods. Surprisingly enough from an economist's point of view, many authors have

*This research was sponsored in part by the Sloan Research Fund, School of Industrial Management, M.I.T. The authors benefited from the advice of M. J. Gordon and Romney Robinson on earlier drafts of the paper.

[1] See for example:

W. W. Cooper and A. Charnes, "Silhouette Functions of Short-run Cost Behavior," Quarterly Journal of Economics, February 1954, pp. 131-150.

K. J. Arrow, T. Harris, and J. Marschak, "Optimal Inventory Policy," Econometrica, July 1951, pp. 250-272.

F. Modigliani and F. E. Hohn, "Production Planning over Time and the Nature of the Expectation and Planning Horizon," Econometrica, January 1955, pp. 46-66.

R. Bellman, "Bottleneck Problems, Functional Equations, and Dynamic Programming" Econometrica, January 1955, pp. 73-87.

A. Dvoretzky, J. Kiefer, and J. Wolfowitz, "The Inventory Problem," Econometrica, April 1952, pp. 187-222, July 1952, pp. 450-466.

H. A. Simon, "On the Application of Servomechanism Theory in the Study of Production Control," Econometrica, April 1952, pp. 247-268.

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H. A. Simon, "Dynamic Programming under Uncertainty with a Quadratic Cost Function," Econometrica, January 1956, pp. 74-81.

E. Schneider, "Absatz, Produktion, und Lagerhaltung bei einfacher Produktion," Archiv für Mathematische Wirtschafts und Sozialforschung, Band IV, Heft I, Leipzig, 1938.

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assumed as given the amounts demanded in each period, and we are hard pressed to find a general treatment in which price as well as all the costs mentioned above are included as variables. This article has the twofold purpose of (a) summarizing the basic notions and problems involved in maximizing profits when demand and cost functions vary in different time periods and when inventory charges and "set-up" costs exist, and (b) indicating an approach for the solution of these problems which uses the more traditional tools of economic analysis. We shall first discuss the problem in a stationary state, i.e., where a constant rate of demand exists over time, in order to explain the nature of the cost variables involved, then present an analysis of the two period case to illustrate the potential use of a presumably familiar technique, and finally offer a method for solving the multi-period case.

I. Stationary State.

Throughout this paper, we shall consider a firm which is producing a single item for sale. In the stationary state, we assume that the firm is confronted with a known demand curve for its product and also a given production function. Since it is our present purpose to illustrate the effect of inventory charges and set-up costs, we are going to assume for the sake of simplicity that production costs are zero; the results of this section can of course be extended to the case of a firm facing a typical cost curve as we demonstrate in subsequent sections. Since the analysis can also be applied to the case of a jobber whose only costs are those of buying and storing and whose revenue is derived from sales, we shall discuss the producer case and the jobber case concomitantly. A very particular kind of "set-up" cost is assumed in this paper, viz., that the firm incur a constant fixed charge if it produces at all, and can avoid such a charge if and only if production is at the zero level.[2] In the case of our jobber, the fixed charge is the cost of acquisition or of placing and receiving the order. Finally we postulate that the inventory carrying charges are a constant amount per item per unit of time in stock. Once again, this assumption can be relaxed, and the reader will see in subsequent sections how we might deal with varying carrying charges.

Consider a three-dimensional graph, with output per "year" measured on the x-axis, dollars on the vertical (z) axis, and the length of time (as a fraction of a "year") between set-ups and production or the placing of orders on the t axis.[3] Examination of a cross-section in zt plane (i.e., holding output constant) will reveal a U-shaped average cost curve. When the time interval is extremely short, production will take place often or many orders will be

[2] Some authors have preferred to consider charges involved with the costs of changing the rate of production between periods and have often assumed that a cost of change is incurred only when the rate of production increases.

[3] For example, if our "year" contains 365 days and t equals 1/5, then production takes place or an order is given every 365/5 = 73 days.

placed, leading to high average costs. On the other hand, when the time interval is long, inventories will be high, leading again to high average costs.[4] For each different level of the yearly amount demanded, there is a different U-shaped curve in the zt plane.

It is very important to note that the higher the carrying costs per unit per year, given set-up costs, the smaller will be the optimal inventory level and consequently the more often production or order placing occurs; similarly, the higher the cost of a set-up, given carrying costs, the larger the inventories and consequently the less often production or ordering takes place.

The minima of each of these average cost curves for given x can be projected onto the xz plane. The locus of these minimum points represents average cost as a function of yearly output, after optimization with respect to the time a lot will last. This average cost curve will be negatively sloped throughout, as economies of scale result from the spreading of set-up costs over an increased number of units.[5] The marginal cost curve corresponding to this average cost curve may be obtained by traditional methods, and the appropriate level of (annual) output may be located by the intersection of the marginal cost and marginal revenue curves. For the optimum level of annual output thus determined, one can readily ascertain the optimal time each lot (obtained by production or acquisition) will last from the cross-section in the zt plane. This time interval determines the number of units per lot, tx. Note that we can equivalently think of the average cost curve as analogous to the traditional long-run envelope curve, where the corresponding "short-run" average cost curves are curves based on a fixed time between orders. Just as different points on the usual long-run average cost curve imply different levels of fixed plant, the points on our long-run average cost curve imply different intervals between set-ups or orders. If our firm increases its output level, in order to minimize average costs at the higher level, it must reduce the time between orders and increase the size of each order.

We may briefly summarize our results by stating that we have exhibited that once inventory charges and set-up costs are introduced, the optimum production and sales pattern for the firm are directly influenced by the levels of these costs. Since in this section we have assumed a stationary rate of demand and costs, we find the firm's optimal pattern is also stationary, i.e., levels of production or amounts ordered as well as the length of the period between production or reordering are constant. When we allow the demand and cost functions to vary over time, we cannot expect our solution to be so simple.

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[4]See F. E. Raymond, Quantity and Economy in Manufacture New York, 1931; and Supra. Let x represent the number of units sold per year, t the fraction of a year between orders, S the cost per order placed, C the unit cost, and I the carrying cost per unit per year. Then, the average cost per unit may be written:

$$C + \frac{S}{tx} + \frac{It}{2}.$$

The derivative of average cost with respect to t is $-S/t^2x + I/2$; the derivative with respect to x is $-S/tx^2$. Thus, for any given time span between orders, we get a continually falling average cost curve, and for any given level of output a U-shaped cost curve.

[5]T. M. Whitin and M. Peston, "Random Variations, Risk, and Returns to Scale," Quarterly Journal of Economics, November, 1954, pp. 610-611. See also preceding footnote.

II. Two Period Case

In introducing the notion of "period" into our analysis, we must be precise in our definition of the associated period demand and cost functions. In the previous section, the choosing of an arbitrary "year" was permissible since we assumed stationary relationships. We now define a period to be that length of time in which all of the following three conditions are satisfied: (a) demand does not change, (b) costs do not change, and (c) only one "set-up" may take place. The first two conditions tend to be market oriented, whereas the last condition tends to be determined by technology. The length of different periods may vary and will influence the inventory charges of carrying a unit from one period to the next; there will be no carrying costs of inventory within a period.

We now assume that the firm faces a bonafide production cost schedule in addition to a demand schedule, set-up and carrying costs in each period. These cost schedules may differ from period to period. (See Figure 1.) We further assume throughout, for the sake of simplicity, that the marginal revenue and marginal cost curves are well behaved, i.e., that marginal revenue curves are at no point upward sloping and are uniquely cut from below by the corresponding marginal cost curves. Let us first consider the case where set-up and inventory costs are zero. The firm is free to store merchandise at the end of the first period. We shall assume the conditions of no initial inventory or inventory at the end of the second period, although the model could as easily handle any other specified levels of initial and closing inventory. Friedrich and Vera Lutz and Weintraub[6] have treated this problem as a price discrimination problem, determining the level of output of the two periods combined by the intersection of the aggregate marginal cost and marginal revenue curves (formed by adding horizontally the marginal cost and marginal revenue curves in each period). The intersections of a horizontal line at the value indicated by the aggregate curves with the marginal cost and marginal revenue curves of each period indicate the outputs and sales levels for each period. Thus, to maximize the present value of profits, the entrepreneur produces OO_1 in period 1, and OO_2 in period 2, and sells OS_1 in period 1 and OS_2 in period 2 at prices OP_1 and OP_2 respectively.

If we introduce a storage cost of i dollars per unit, we may lower the second period marginal revenue and cost curves by the amount i and proceed as above to derive aggregate curves and corresponding output and sales levels;[7] this particular procedure for treating inventory charges is thus analogous to that which is sometimes used in treating transportation costs between spatial markets.

If the diagrams in Figure 1 for the two periods were in reverse order, then the solution would lead to the holding of "negative inventories" in the new period 1. Whereas with spatial markets, we are free to place the diagrams in any order, with time periods, we can observe only one particular sequence. If negative inventories occur, then we must apply the simple monopoly solution to each period. Although the modification of the "price-discrimination"

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[6] F. and V. Lutz, The Theory of Investment of the Firm, Princeton, 1951 pp. 94-96; S. Weintraub, Price Theory, New York, 1949, pp. 388-393.

[7] If carrying costs are not strictly proportional to the number of units of inventory, then recourse to a calculus model will be necessary.

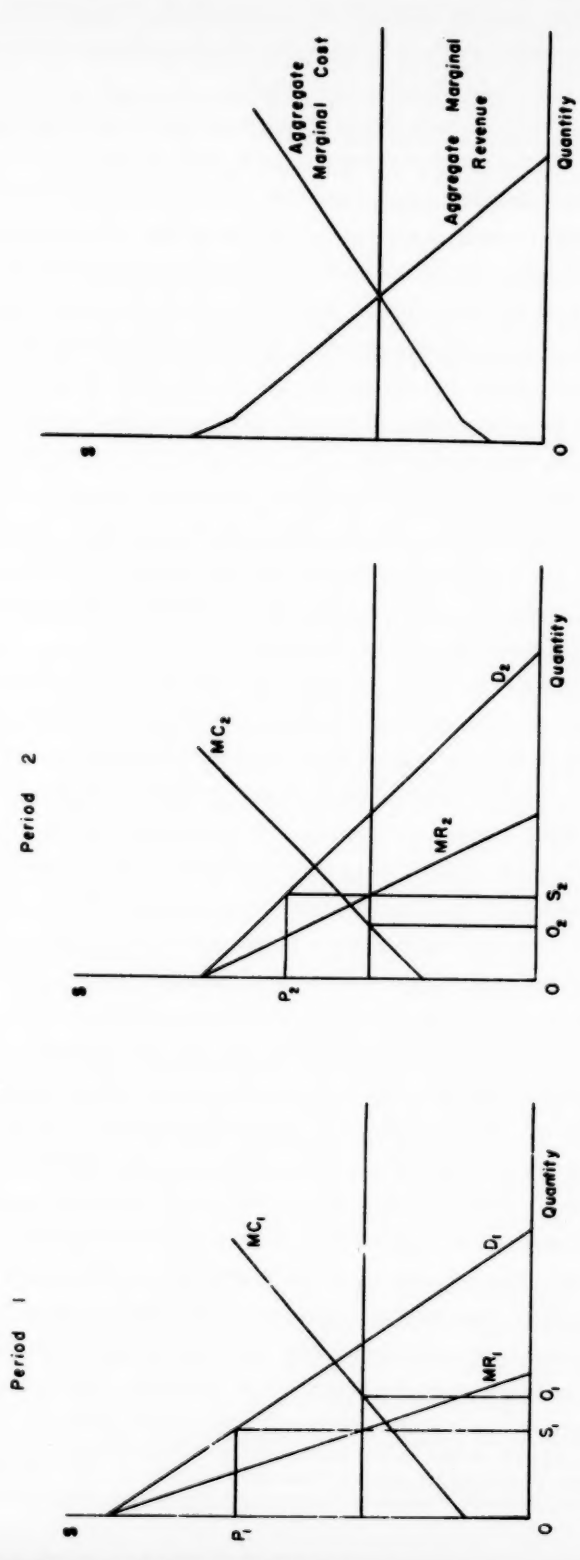

Aggregate Marginal Cost
Aggregate Marginal Revenue
Quantity
$
0
Period 2
$
P_2
MC_2
MR_2
D_2
O_2 S_2
Quantity
0
Period 1
$
P_1
MC_1
MR_1
D_1
S_1 O_1
Quantity
0

Figure 1

solution, which arises when negative inventories are indicated, is simple for the two period case, the needed optimum modification in a multi-period analysis is not so clear, and we shall need to deal with the solution in detail in the next section.

We should note that in our model, being analogous to price discrimination, the intersection of marginal cost and marginal revenue curves in each period is of no special significance. The important fact is that the values (after taking account of inventory charges) of marginal cost and marginal revenue are equated. Furthermore, in contrast to price discrimination models, our solution depends on the particular time ordering of the periods.

Let us now introduce set-up charges. Previously the costs being analyzed had a "continuity" property (that is, a small change in output or inventory involved only a small change in production costs or carrying costs); set-up charges on the other hand are of a lump-sum or sunk-cost nature and the usual procedure of "equating at the margin" must be somewhat modified. As a first suggestion, we might consider the approach of trying every possible distribution of set-ups over the periods and picking the best: (produce in period 1, produce in period 2), (produce in period 1, not produce in period 2), (not produce in period 1, produce in period 2), (not produce in period 1, not produce in period 2). In our example, clearly only the first two possibilities warrant investigation. If the set-up cost in period 1 and/or period 2 were sufficiently high, all four cases would have to be examined.

When we solve for the case (produce in period 1, produce in period 2), we use the construction of Figure 1 with carrying charges i. Total profits equal total revenue less costs of production less inventory charges less two set-up charges. The solution for (produce in period 1, not produce in period 2) involves an aggregate marginal revenue curve for both periods (the period 2 curve being lowered by i) and a single marginal cost curve which is identically the marginal cost curve for period 1. Total profits here are equal to total revenue less total cost of production less inventory charges less one set-up charge. Of the solutions for the two cases, the optimum production and sales pattern is that which yields the maximum profits. High set-up charges, for example, would make it profitable to produce in the first period only, while low set-up charges make it more profitable to produce in both periods.

Before going to the n-period case, we might note that our approach can be extended to apply to situations where the demand in the second period is not independent of the first period price and quantity decisions. For example, for each price charged in the first period we know the marginal revenue for the first period and we assume there is a single associated demand curve for the second period. The horizontal distances for the marginal revenue of the first period and the marginal revenue corresponding to the appropriate demand curve for the second period may then be added. The intersection of the aggregated marginal cost and marginal revenue curves will determine total output. The prices and output of each period may then be ascertained by the intersection of a horizontal line with the marginal cost and appropriate marginal revenue curves of each period at the height of the crossing of the aggregate curves. This approach seems simpler than those recommended by some other authors.[8]

[8] M. Reder, "Intertemporal Relations of Demand and Supply within the Firm," Canadian Journal of Economics and Political Science, February 1941, pp. 25-38; M. J. Bailey, "Price and Output Determination; Related Products," American Economic Review, March 1954, pp. 83-93.

The results of this section have demonstrated first that inventory charges can be handled in a manner analogous to price discrimination and/or transportation cost problems with the important proviso that the time sequence of the periods may lead to inventory infeasibilities if a simple price discrimination model is solved. Secondly we have seen that set-up charges seem to necessitate some type of enumerative procedure in determining the true optimum pattern.

III. n-Period Case (With Example for n = 3)

The solution to the general n-period case is indeed complicated and hence it will be to our advantage to consider first certain n-period cases in which simplifying assumptions are made.

It is easy to arrive at the following feasibility relationships: (a) total production up to and including any period must be at least as great as total sales,[9] (b) production and sales in each period must be non-negative quantities, and as a result (c) inventory at the beginning of any period which equals total production minus total sales of all previous periods is non-negative. A solution for the entire n-periods by calculus, using Lagrangian multipliers, is only admissible provided that all set-up costs are zero and that the resulting solution satisfies relations (a) and (b) above; as we have seen even in the two period case where the periods in Figure 1 were reversed, in a calculus (or price discrimination) solution, condition (a) may easily not be satisfied because of the particular time sequence of the periods. Let us alternatively consider a graphical approach analogous to that used in Section II under the assumption that set-up costs are zero; the analysis applies to the cases of both zero and non-zero inventory charges.

As before, we construct by horizontal addition of all the marginal curves, aggregate marginal revenue and marginal cost curves over the n periods, and solve for the resulting production and sales in each period. Note that if we are considering a constant positive inventory carrying charge i between all periods, then we shift the curves for period 2 down by the amount i (as before), the curves for period 3 down by 2i, the curves for period 4 down by 3i etc. If the periods are of different length, then i will differ as between periods and the curves must be shifted downward accordingly. For example, the curves for period 3 are shifted downward by the sum of the inventory charges between periods 1 and 2, and between periods 2 and 3. This shifting preserves the correct inventory charge relationships between all periods. If condition (a) above is satisfied (condition [b] will automatically be satisfied by our construction), we have then found our optimal solution. But if an infeasibility arises, failure in condition (a), we must alter the pattern indicated.

Suppose t' is the last period in which an infeasibility exists (i.e., a "negative inventory" level); total production must consequently exceed total sales over periods t', $t' + 1$, ..., n, since our construction assures us that total production over all n periods equals total sales. We try a new solution by partitioning the n periods into two groups, those before and including t', and those after t'. The standard procedure is applied to the former group of periods

[9] Recall we are assuming no initial inventory.

and the feasibility conditions are examined again. If feasibility is obtained, the aggregating technique is next applied to the latter group of periods, and provided feasibility also holds for this latter group, we have arrived at the optimal pattern over the n periods. If under the partitioning feasibility still does not exist in the first group of periods, this first group is itself partitioned as before. The process continues until finally a group of periods 1, 2, ..., t'' admits a feasible solution. Then the suggested aggregating technique is applied to the group of remaining periods $t'' + 1$, ..., n; at this stage the partitioning procedure may also be necessary.

If for example we consider the firm's facing identical constant (horizontal) cost curves and a positive inventory charge i, the first application of our aggregating technique would indicate that all production should take place in the last period. This result has an interesting interpretation if we think of the possibility of the firm's taking orders over n periods, earning interest on the revenues received from the sales, and delivering at period n; clearly, it is to the firm's advantage to withhold production as long as possible so as to maximize interest earnings. But disallowing future deliveries, we go through the partitioning technique (which in this case leads to the maximal number of partitions) and finally arrive at the optimal feasible pattern of sales and production: produce in each period only for sales in that period and carry no inventories; the result is obvious of course since the cost of production is the same in all periods, whereas inventory charges are positive.

In general when there are no set-up costs, if for any period t''' we find that no inventory is being carried to period $t''' + 1$, we may conclude that if demand and/or production costs increase some in periods up to and including t''', or if demand and/or production costs decrease in periods after t''', there still will be no carry over of inventories from t''' to $t''' + 1$, although, of course, sales and production patterns alter within the group of periods where the changes occur; in other words, the two groups of periods are independent of each other for certain broad types of changes in demand and costs.

Once we drop the assumption that set-up costs are zero, both the quantitative and qualitative aspects of our previous conclusions change. The aggregate marginal curve approach must be modified in order to take into consideration the problem of discrete jumps in the profit function at certain levels of output and sales.[10] Here we turn to an enumerative method. There are at most a finite number of possible configurations of set-up costs in our example. Hence, one approach might be to enumerate all cases and choose the optimal one. But this approach, which we actually used in the two period model, is not helpful except for problems of trivial size, since there are 2^n possible cases involving different arrangements of set-up costs.[11] Certainly for n greater than about 10, this procedure is impractical unless one uses automatic computing equipment and even then it is somewhat questionable.[12] Some computational algorithm is

[10] Without set-up costs, linear programming techniques can also solve this type of problem where the profit function is convex. However, there is no general technique in linear programming to handle the set-up cost problem.

[11] In each period there are two possibilities, i.e., to produce or not produce.

[12] We should note that this exhaustive suggestion does demonstrate that theoretically a procedure exists which will yield the optimum pattern, and for small n, it is actually better than the one which we shall give below. It should be clear that, for any particular one of the 2^n possibilities, we must also employ the aggregating and partitioning techniques described above.

needed which is not dependent on complete enumeration, but enables a systematic approach to optimality;[13] we shall offer such a solution below.

We shall first consider a case with very simplifying assumptions, namely that the marginal cost curve is horizontal and identical in all periods. This case will give us a basic understanding of underlying difficulties which lie ahead. It is clear that the firm will not reduce its production costs by producing in an earlier period for sales in a later period. Under our assumptions, the cost balancing arises only between carrying costs and set-up charges.[14] If the firm produces at all in a given period, no inventory will enter that period, for there is nothing to be gained by bringing in inventory, whereas carrying charges are incurred by so doing. This property of independence between the periods enables us to reduce the computational problem significantly. First, we can solve for the optimal production, sales, and profits in period one, given just the period one demand and cost curves.[15] When period two is added, we need consider only two possibilities: (a) produce in period one for period one and produce in period two for period two; and (b) produce in period one for both periods. Under (a), we utilize the optimum solution for period one and a similar optimum for period two. Under (b), we use the aggregated marginal revenue curves[16] and the (constant) marginal cost curve to determine optimal sales, production and profits. We select the more profitable alternative, i.e., (a) or (b). When period three is added, we have three alternatives: (a) produce in period three; (b) produce in period two for periods two and three; and (c) produce in period one for all three periods. For (a), we solve the period three production and sales pattern with reference to period three data only, combining the resulting profits with the optimal profits for the first two periods. For (b), we use the two period technique for periods two and three, and combine the resulting profits with the optimal pattern for period one. For (c), we use a three period aggregative technique using the three marginal revenue functions, (appropriately adjusted for carrying charges) and the (constant) marginal cost. Next, we select the most profitable alternative among (a), (b), and (c). Continuing, we find that we have only n alternatives to consider in the nth period. Table 1 below presents the essential aspects of the technique in schematic form. The most profitable pattern in any column is indicated by the asterisks. For example, $(1, 2, 3)^*$ represents the optimum policy for periods one, two, and three, which could be (a), (b), or (c) above. The underlined combinations imply that production for all periods in the combination takes place in the first period underlined. The n possible patterns in the nth period, would be $(1, 2, 3, 4, \ldots, n-1)^*$ $\underline{n}$; $(1, 2, 3, 4, \ldots, n-2)^*$ $\underline{n-1, n}$; ..., $\underline{1\ 2\ 3\ 4 \ldots n}$. From these n patterns is selected the optimal pattern $(1, 2, 3, \ldots n).^*$

Other important computational simplifications are possible. Whenever the optimal pattern over periods 1 through t includes two set-ups, all periods prior to the last set-up may be ignored in determining the optimal policy in future periods. For example, if the optimal pattern involved setting up in periods 1 and 2, period 1 no longer influences the optimal pattern

[13] In linear programming, the "simplex method" is an example of such an algorithm.

[14] The reader will note we are discussing a generalized version of the model in Section I.

[15] If the set-up cost is high, the solution may indicate zero levels for production and sales.

[16] In aggregating, it must be kept in mind that the second period revenue curve must be lowered by the amount of unit carrying charges.

TABLE 1

Period	1	2	3	4
Possibilities	1	(1)* 2 1 2	(1, 2)* 3 (1)* 2 3 1 2 3	(1, 2, 3)* 4 ... (1, 2)*3 4 ... (1)* 2 3 4 ... 1 2 3 4 ...
Optimal	(1)*	(1, 2)*	(1, 2, 3)*	(1, 2, 3, 4)*

for period 3 and later periods. The reason for this is intuitively clear. If carrying charges are sufficiently high to justify set-ups for both periods 1 and 2 when considered without reference to later periods, then only loss can result from any subsequent reduction in the number of set-ups. Table 1, therefore, greatly overstates the computational problem, even though it in itself constitutes a drastic simplification when compared to complete enumeration. The independence property, i.e., the condition that once production takes place in a given period, the optimal pattern from that period on is independent of the sales and production of previous periods, has played a key role in the above simplifications. It has enabled us to disregard periods previous to a period in which a set-up occurs and to use a "forward-type" of solution.[17] Once we drop the assumption of identical and horizontal cost functions in each period, the independence property vanishes and we must resort to different optimizing techniques. Inventories entering a given period may consequently turn out to be produced in several preceding periods, making it impossible to use a "forward-type" solution. We now turn to the general case.

The approach that underlies our method for solving the general case has been provided by Dvoretzky, Kiefer, and Wolfowitz.[18] We are extending their analysis by explicitly considering price as a variable. The essential feature of their approach is that for any given "net inventory" position (defined as initial inventory minus final inventory in a period), it is a simple matter to find the optimum values for sales and production. The interpretation of net inventory makes this fact obvious. If net inventory is positive or zero, then this amount of goods is available for sale in the period without incurring any production or set-up costs. If net inventory is negative, then this amount must be produced during the period for use in later periods, and a given amount of production costs and the set-up costs are inevitable; any production above this minimal level is directly for current sales.

Specifically, given both the entering and leaving inventory levels in the first period, we can determine the optimum amount of sales and production for the first period. Total profits over all periods in the first instance will be a function of the profit in period 1 and the profits accruing in later period given the specific amount of inventory leaving period 1. Hence, we could easily determine the optimum profit level over the n periods if we could ascertain the profits associated with each level of inventory leaving period 1. But these profits in turn are a function of the sales-production-net inventory position of period 2. Given the amount of

[17] By this we merely mean moving from present to future periods.

[18] "The Inventory Problem," Econometrica, April, 1952, pp. 187-222, and July 1952, pp. 450-466.

inventory entering period 2, we consider each amount of inventory leaving period 2, and from the net inventory position determine period 2 profits. If we could know the profits associated with the inventory leaving period 2, we could determine profits over periods 2 to n. As the reader may see, we can continue our breakdown of profits into period profits and profits associated with outgoing inventories for periods 3, 4, ..., and finally period n. But in period n, our last period, outgoing inventory profits may be calculated easily. If we require no ending inventories, there are no outgoing inventory profits; if we require a certain amount of ending invetories, we assign an arbitrary value to them. If we permit any positive amount of ending inventory, then the actual assigned value is important.

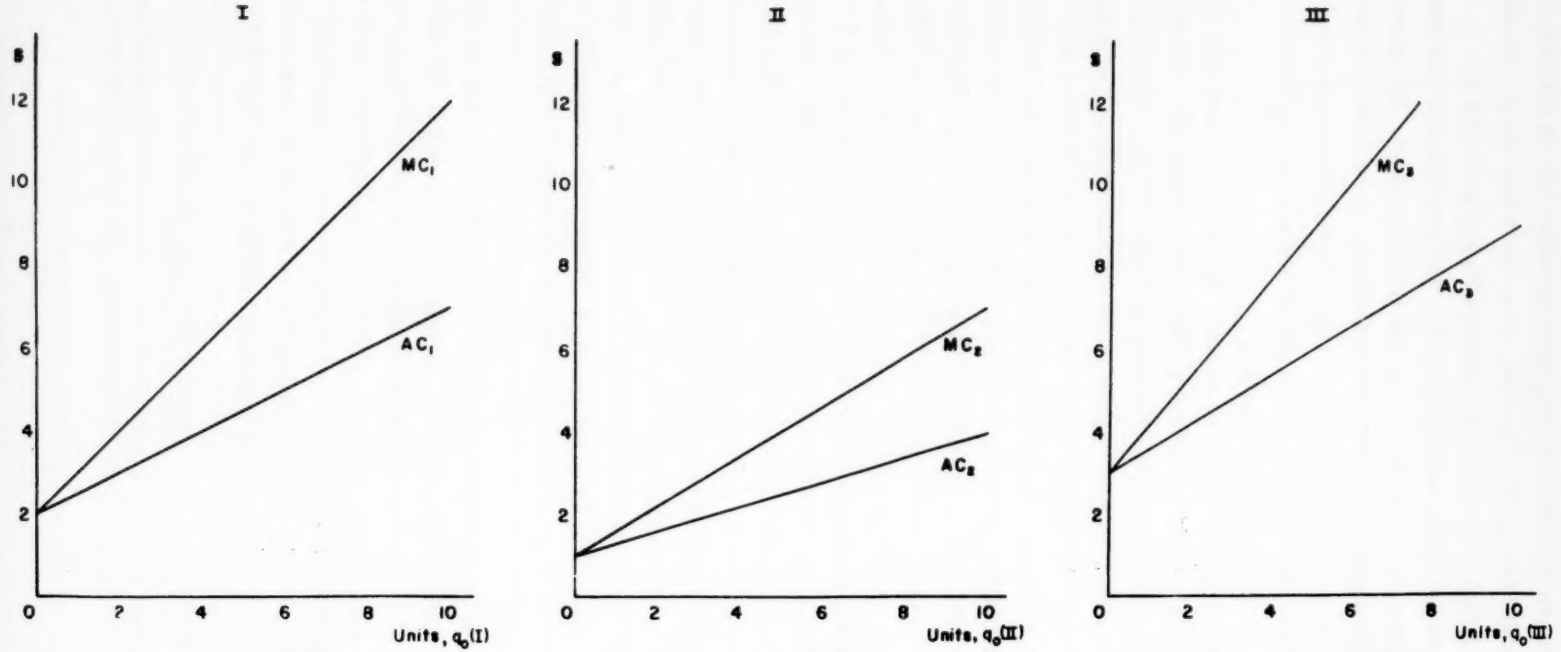
Our path towards a solution now becomes evident. We determine profits in period n for each possible amount of inventory entering period n. We can next determine, for each possible amount of inventory entering period n - 1, profits over periods n - 1 and n combined. Each possible period n - 1 net inventory position implies that profits are determinate in period n - 1 and also in period n.[19] We then go on to period n - 2, combining n - 2 profits with the profits for the (already combined) periods n - 1 and n, etc. Finally, when we arrive at the first period, we have derived the information which we desire, viz., the profits associated with each level of outgoing inventory for period 1. Period 1 differs from all the other periods in that the amount of initial inventory is known. For any period t after the first, there are alternative levels of entering inventory, thus giving several "best" patterns over periods t through n, each pattern corresponding to one of the alternative levels of inventory entering period t. Since there is an unique level of inventory entering the first period, we can consider the alternative levels of outgoing inventory to period 2, determine the corresponding profit level for periods 1, add the previously computed associated profit level for periods 2 through n, and as a result ascertain the optimal profit level and pattern or production and sales over all periods.

Given a specific set of demand and cost functions, we could solve for the optimum by the use of calculus and a somewhat complex compounding of formulas for obtaining constrained maxima. What we wish to exhibit here is a method which considers discrete points on the demand and cost functions. We shall make use of a three period example to illustrate the technique (see Figures 2 and 3).[20]

We first construct Tables II and III yielding the total cost and total revenue for different amounts of production and sales. These tables are simply derived from the cost and revenue data summarized in Figures 2 and 3. Although the tables are potentially "infinite" in length, we have certain a priori bounds on sales and hence production in each period. As is well known, we need never consider sales beyond the point of unit elasticity on the demand curve. We can derive the limits on the table of costs, since production in the first period need never be larger than the amount of total sales at unit elasticity of demand over all periods, period 2 production

[19] Incoming inventory in period n - 1 minus net inventory in period n - 1 equals outgoing inventory for period n - 1 equals incoming inventory for period n.

[20] For this simple example it may be easier to use the exhaustive technique of enumerating all $8(=2^3)$ possible combinations as we have metnioned above. We have selected such a small value of n to conserve space.



$AC = 2 + \frac{1}{2} q_0(I)$

$MC = 2 + q_0(I)$

$AC = 1 + \frac{3}{10} q_0(II)$

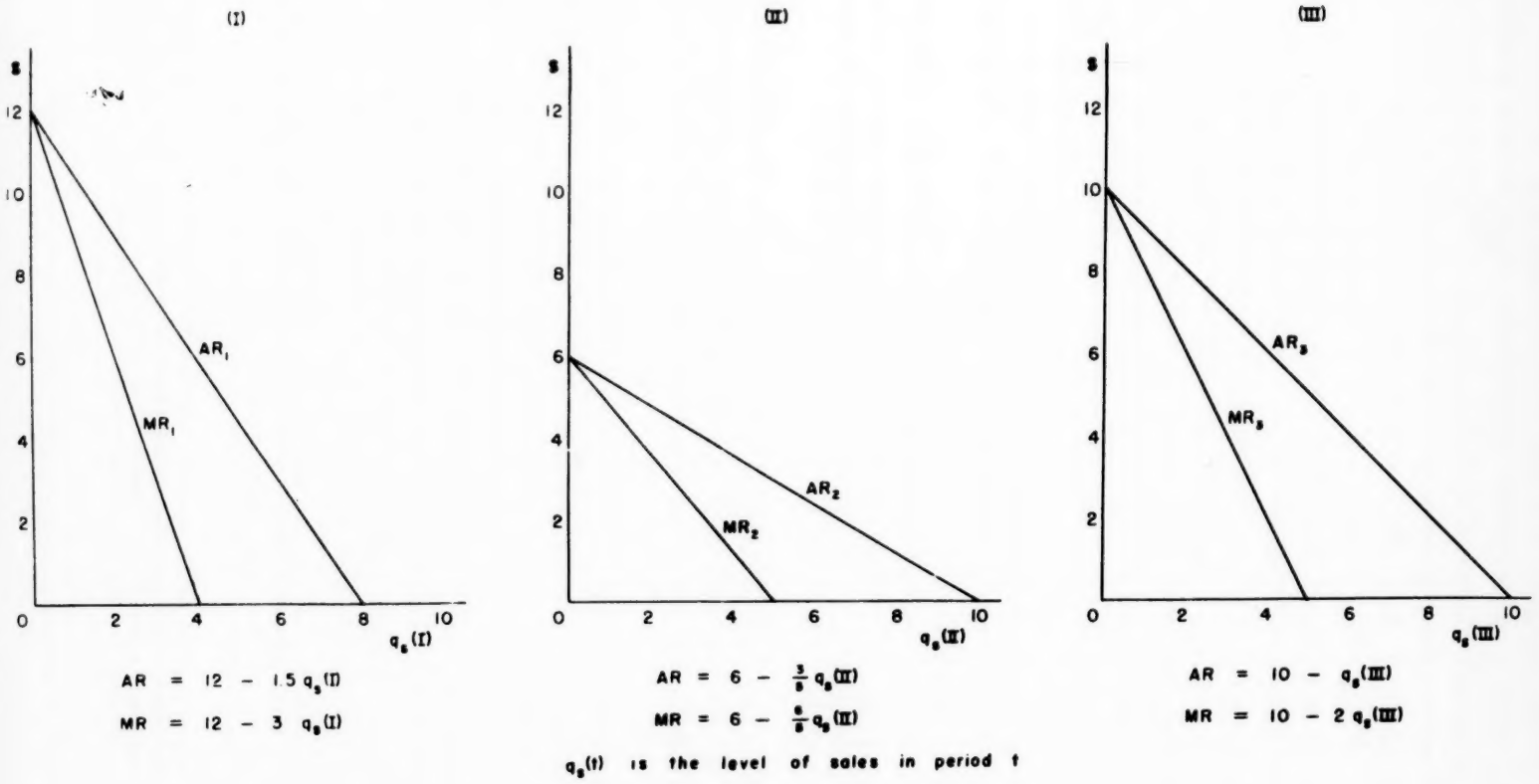
$MC = 1 + \frac{3}{5} q_0(II)$

$AC = 3 + \frac{3}{8} q_0(III)$

$MC = 3 + \frac{6}{8} q_0(III)$

$q_0(t)$ is the level of production period t.

Figure 2



$q_0(t)$ is the level of production period t; $q_s(t)$ is the level of sales in period t.

Figure 3

TABLE II

TC(t)

$q_o(t)$	t = I	II	III
1	2.5	1.3	3.6
2	6.0	3.2	8.4
3	10.5	5.7	14.4
4	16.0	8.8	21.6
5	22.5	12.5	30.0
6	30.0	16.8	
7	38.5	21.7	
8	48.0	27.2	
9	58.5	33.3	
10	70.0	40.0	
11	82.5		
12	96.0		
13	110.5		
14	126.0		

($q_o(t)$ = production in period t.)

TABLE III

TR(t)

$q_S(t)$	t = I	II	III
1	10.5	5.4	9.0
2	18.0	9.6	16.0
3	22.5	12.6	21.0
4	24.0	14.4	24.0
5		15.0	25.0

($q_S(t)$ = sales in period t.)

need never be larger than the amount of total sales at unit elasticity of demand over periods 2 through n, etc. If there is an upper limit to capacity which is below the potential amounts of production just given, then an infinite cost is assigned to all amounts of production above capacity.

For each period we construct a set of tables to cover the possible eventualities. The rows of each table represent incoming inventory, and the columns represent outgoing inventory.[21] In our three period example, tables for the first period will have only one row, representing the given amount of initial inventory. Tables for the last period will have only one column, representing the desired amount of ending inventory.[22] For each period, starting at the last period, we compile two profit tables, one on the assumption that set-up costs are incurred in the period, and another on the assumption that they are not incurred. To derive these tables we must construct 5 subsidiary tables (for each period):

[21] In our presentation we are constructing tables to cover all conceivable cases by using as upper bounds the total quantities that could be sold at unit elasticity of demand. In most specific cases, one will find that these tables are larger than necessary. Furthermore, "units" of the commodity can be assigned in such a manner as to reduce the number of rows and columns to manageable size.

[22] It is possible to generalize the approach to allow different levels of ending inventories in the last period. However, for ease of exposition we shall consider only the case where there is no value attached to these inventories. Thus we need consider only the case of zero last period ending inventory. Similarly we start arbitrarily with zero inventory at the beginning of the first period.

t-(1) Total Revenue – Total Production Costs (i.e., exclusive of set-up costs and carrying charges), assuming production takes place. Optimal levels of production and sales can be ascertained for each level of incoming inventory from marginal cost and marginal revenue considerations.[23] Furthermore, since only the net inventory position is important in determining the level of sales and production, all entries on northwest-southeast diagonals are the same.

t-(2) Total revenue from sales of positive net inventory, assuming no production takes place. All entries above the main diagonal may be ignored, since negative net inventory is implied requiring production, contrary to our assumption. As in t-(1), equality holds for all of the entries along the northwest-southeast diagonals.

t-(3) Inventory carrying charges. A table of inventory carrying charges can be readily constructed for each level of incoming inventory. These can behave in as simple or as complicated a manner as we choose.

t-(4) Set-up costs.

t-(5) Profits associated with each level of outgoing inventory under optimal policy over subsequent periods. (For the last period we shall assume in our example that these profits are zero.)

We obtain profit table t-A, under the assumption that production takes place, by adding tables (1) and (5) and subtracting tables (3) and (4). We derive profit table t-B, under the assumption that production does not take place, by adding tables (2) and (5) and subtracting Table (3).

We now form table t-C, representing the optimum patterns, by selecting for each level of incoming inventory the largest profit figure in tables t-A and t-B; associated with this maximal figure is the corresponding optimal level of sales and production. Figure 4 gives the tables for period III in our example. If, for example, incoming inventory for this period is 1 unit, profits of 8.6 are obtained by producing 2 units and selling 3; if incoming inventory is 3 units, profits of 18 are obtained by selling 3 and producing 0 etc. The procedure is then to work back toward the first period calculating optimal production and sales for each level of incoming inventory in periods II and III (combined) and finally for periods I, II, and III (combined). We can then select the largest profit figure for the three periods. The outgoing inventory of period I is carried over to period II as incoming inventory. The largest profit figure for this incoming inventory is written in II-C, which in turn determines the corresponding production, sales and outgoing inventory for period II. The latter is the incoming inventory of period III and determines the optimal period III production and sales levels. In this manner we move forward selecting the optimal levels of production and sales period by period for all periods.

[23] A simple graphical technique for determining the sales and production for a given net inventory position consists of drawing the period's MC and MR curves on separate pieces of translucent graph paper. For a zero level of net inventory, we superimpose the axes of the two curves and determine the intersection point. (In this case sale = production.) For a net inventory of +I, we shift the marginal revenue curve one unit to the left and again determine the point of intersection. In this case total sales exceeds total production by one unit, etc.

TABLE	III-(1)	III-(2)	III-(3)	III-(4)	III-(5)	III-A	III-B	III-C		
Outgoing Inventory	TR-TC	TR	Carrying Charges at $1 per unit	Set-up Costs $3	Outgoing Inventory Profits	Positive Production	No Production	Optimum		
								Profits	Sales	Production
	0	0	0	0	0					
Incoming Inventory 0	7.6	0	0	3	0	4.6	0	4.6	2	2
1	12.6	9	1	3	0	8.6	8	8.6	3	2
2	17.4	16	2	3	0	12.4	14	14	2	0
3	21.0	21	3	3	0	15	18	18	3	0
4	24.0	24	4	3	0	17	20	20	4	0
5	25.0	25	5	3	0	17	20	20	5	0

Figure 4

TABLE	II-(1)						II-(2)						II-(3)						II-(4)					
Outgoing Inventory	TR-TC						TR						Carrying Charges						Set-up Costs					
Incoming Inventory	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5
0	6.9	3.9	.8	-2.9	-7.1	-11.4	0	*	*	*	*	*	0	0	0	0	0	0	3	3	3	3	3	3
1	9.4	6.9	3.9	.8	-2.9	-7.1	5.4	0	*	*	*	*	1	1	1	1	1	1	3	3	3	3	3	3
2	11.3	9.4	6.9	3.9	.8	-2.9	9.6	5.4	0	*	*	*	2	2	2	2	2	2	3	3	3	3	3	3
3	13.1	11.3	9.4	6.9	3.9	.8	12.6	9.6	5.4	0	*	*	3	3	3	3	3	3	3	3	3	3	3	3
4	14.4	13.1	11.3	9.4	6.9	3.9	14.4	12.6	9.6	5.4	0	*	4	4	4	4	4	4	3	3	3	3	3	3
5	15.0	14.4	13.1	11.3	9.4	6.9	15.0	14.4	12.6	9.6	5.4	0	5	5	5	5	5	5	3	3	3	3	3	3
6	x	15.0	14.4	13.1	11.3	9.4	x	15.0	14.4	12.6	9.6	5.4	x	6	6	6	6	6	x	3	3	3	3	3
7	x	x	15.0	14.4	13.1	11.3	x	x	15.0	14.4	12.6	9.6	x	x	7	7	7	7	x	x	3	3	3	3
8	x	x	x	15.0	14.4	13.1	x	x	x	15.0	14.4	12.6	x	x	x	8	8	8	x	x	x	3	3	3
9	x	x	x	x	15.0	14.4	x	x	x	x	15.0	14.4	x	x	x	x	9	9	x	x	x	x	3	3
10	x	x	x	x	x	15.0	x	x	x	x	x	15.0	x	x	x	x	x	10	x	x	x	x	x	3

* Involve accumulation of inventory, contrary to assumption of no production.
x Sales above quantity at unitary demand elasticity.

Figure 5

TABLE	II-(5)						II-A						II-B						II-C		
Incoming Inventory \ Outgoing Inventory	Outgoing Inventory Profits						Positive Production						No Production						Optimum		
	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5	Profits	Sales	Production
0	4.6	8.6	14	18	20	20	8.5	9.5	11.8	12.1	9.9	5.6	4.6	*	*	*	*	*	12.1	2	5
1	4.6	8.6	14	18	20	20	10.0	11.5	13.9	14.8	13.1	8.9	9.0	7.6	*	*	*	*	14.8	2	4
2	4.6	8.6	14	18	20	20	10.9	13.0	15.9	16.9	15.8	12.1	12.2	12.0	12.0	*	*	*	16.9	2	3
3	4.6	8.6	14	18	20	20	11.7	13.9	17.4	18.9	17.9	14.8	14.2	15.2	16.4	15.0	*	*	18.9	3	3
4	4.6	8.6	14	18	20	20	12.0	14.7	18.3	20.4	19.9	16.9	15.0	17.2	19.6	19.4	16.0	*	20.4	3	2
5	4.6	8.6	14	18	20	20	11.6	15.0	19.1	21.3	21.4	18.9	14.6	18.0	21.6	22.6	20.4	15.0	22.6	2	0
6	x	8.6	14	18	20	20	x	14.6	19.4	22.1	22,3	20.4	x	17.6	22.4	24.6	23.6	19.4	24.6	3	0
7	x	x	14	18	20	20	x	x	19.0	22.4	23.1	21.3	x	x	22.0	25.4	25.6	22.6	25.6	3	0
8	x	x	x	18	20	20	x	x	x	22.0	23.4	22.1	x	x	x	25.0	26.4	24.6	26.4	4	0
9	x	x	x	x	20	20	x	x	x	x	23.0	22.4	x	x	x	x	26.0	25.4	26.0	5	0
10	x	x	x	x	x	20	x	x	x	x	x	22.0	x	x	x	x	x	25.0	25.0	5	0

* Involve accumulation of inventory, contrary to assumption of no production.
x Sales above quantity at unitary demand elasticity.

Figure 5 (continued)

The entries in Table II-C represent the optimal profits for periods II and III for each level of incoming inventory to period II. We proceed to the first period in which we assume that there is no incoming inventory. Therefore, we need consider only one row in each table, Figure 6.

The optimal policy for period I is to earn the maximum profit figure of 21.1. This figure results from producing two units in the period and selling them both. There is therefore no incoming inventory in the second period. The optimal policy for period II is to produce 5 units and sell 2, giving a 3 unit incoming inventory in the last period. The optimal policy in the final period is to sell the 3 units of stock, refraining from production. Thus we have one period with sales and production matched, one period of inventory accumulation, and one period of zero production and inventory decumulation.

A comparison of different optimal production-sales-inventory patterns under varying assumptions is given in Table IV. All of these cases assume the revenue and direct production cost functions of Figures 2 and 3. Case 6 summarizes the model discussed above. In Case 1 only we relax the assumption of a fixed sequence of time periods. Here we have a situation of spatial monopoly with price discrimination, no set-up costs, and no carrying charges and obtain a pattern quite different from our other models in that negative outgoing inventories are permitted. In the optimal production pattern, inventories will be accumulated in the second "period" and decumulated in periods one and three.

In the case where there are neither carrying charges nor set-up costs (Case 2) the main difference between the production pattern and that of Case 6 is that the absence of set-up costs makes it profitable to produce in period III. If carrying charges ($1 per unit/period) are interjected into the situation, (Case 3), the effect is, as one would expect, to reduce the amount of carry over from period II to period III. Case 4 represents a case where set-up costs are $3.00 but carrying charges are zero. It is exactly like our original example (Case 6) except that there are no carrying charges and considers only discrete units of production, inventory, and sales. The absence of carrying charges, if it is to affect the production pattern, will reduce the number of set-ups. In the problem at hand, the magnitude is too small to change the optimal production program from that of Case 6. The net effect of zero carrying charges is to save costs of $3 ($1/unit for 3 units) for carrying inventory from period II to III.

If continuous functions are considered, it is possible to apply the price discrimination technique (assuming divisibility), allowing production to take place only in the periods indicated in the discrete solution. The presence or absence of carrying charges now exerts an influence. With zero carrying charges (and $3 set-up cost) it is profitable to carry over 3.06 units from period II to III (Case 5). With $1 (per unit per time period) carrying charges and $3 set-up costs, it is profitable to carry over only 2.64 units (Case 7). Provided that the discrete points are sufficient in number, the marginal curves are well behaved, and the "next best" discrete solution results in strictly less total profits, the discrete solution will give the correct solution to the set-up problem, and the price discrimination approach will give an exact solution to

the production, sales, and inventory problems. It can be shown that as the number of periods increases, the number of computations involved in the approach outlined above does not exhibit the rapid exponential rise of enumerative techniques.[24]

These cases demonstrate the tendency of carrying charges to reduce the levels of inventories being carried, of set-up costs to eliminate production in some periods, and of different cost assumptions to result in different optimal patterns.

IV. Summary and Conclusions

We have presented a theoretical analysis of the dynamic problems of a firm faced with varying demands and production costs over time; we have seen that difficulties arise because of the particular time ordering of the periods and because of the lumpiness of certain costs. Although our purpose was not to give a taxonomic treatment of such problems, we have analyzed in some detail those cases which we believe are useful for indicating the inherent difficulties. Furthermore, we have presented a technique which in our opinion is efficient for the solution of particular multi-period models. Finally, we have attempted to combine traditional tools of economic analysis with some of the newer algebraic and statistical approaches so as to give the economist a not too unfamiliar treatment of certain dynamic problems in the theory of the firm.

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[24] As a rule of thumb, the above technique yields a significant reduction in computations over the completely exhaustive (2^n) approach when the average number of discrete points on the period marginal curves is significantly less than $2^{n+1}/n$.

TABLE	Incoming Inventory			Outgoing Inventory 0	1	2	3	4	5	6	7	8	9	10
	I-(1)	TR-TC	0	12	7.5	2	-4.5	-12	-19.5	-28	-37.5	-48	-58.5	-70
	I-(2)	TR	0	0	*	*	*	*	*	*	*	*	*	*
	I-(3)	Carrying Charges	0	0	0	0	0	0	0	0	0	0	0	0
	I-(4)	Set-up Costs	0	3	3	3	3	3	3	3	3	3	3	3
	I-(5)	Outgoing Inventory Profits	0	12.1	14.8	16.9	18.9	20.4	22.6	24.6	25.6	26.4	26	25
	I-A	Positive Production	0	21.1	19.3	15.9	11.4	5.4	.1	-6.4	-14.9	-24.6	-35.5	-48
	I-B	No Production	0	12.1	*	*	*	*	*	*	*	*	*	*

* Involve accumulation of inventory, contrary to assumption of no production.

Figure 6

I-C		
Optimum		
Profits	Sales	Production
21.1	2	2

Figure 6 (continued)

TABLE IV*

Cases	1	2	3	4	5	6	7
Profit	31.23	30.90	28.77	24.10	24.65	21.10	21.80
q_o(I)	1.90	2.50	2.50	2.00	2.50	2.00	2.50
q_s(I)	2.70	2.50	2.50	2.00	2.50	2.00	2.50
Outgoing Inventory (I)	-.80	0	0	0	0	0	0
q_o(II)	4.84	4.49	3.91	5.00	4.82	5.00	4.54
q_s(II)	1.74	1.92	2.21	2.00	1.76	2.00	1.90
Outgoing Inventory (II)	3.10	2.57	1.70	3.00	3.06	3.00	2.64
q_o(III)	.75	.58	1.12	0	0	0	0
q_s(III)	3.05	3.15	2.82	3.00	3.06	3.00	2.64
Outgoing Inventory (III)	-2.30	0	0	0	0	0	0

*Figures are rounded.

LOSS RESULTING FROM THE USE OF INCORRECT DATA IN COMPUTING AN OPTIMAL INVENTORY POLICY

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Extensive work has been done in deriving the optimum inventory policy for various models based on different assumptions. The following is an attempt to find the cost consequences for two such models of executing a policy computed from erroneous data. Part I treats a case of certain demand. Part II treats a case of demand being a uniformly distributed random variable.

I. CASE OF CERTAIN DEMAND

In the following, an algebraic expression will be derived giving the increase in total variable cost as a function of the error in the data used in computing the optimal inventory policy.

T. M. Whitin [Ref. 2, pp. 32-33] gives the formula for total variable cost and derives the optimum purchase quantity as follows:

(1) $$TVC = \frac{QC}{2} I + \frac{Y}{Q} S$$

where TVC = total annual variable costs
Q = purchase quantity
C = unit cost
S = procurement expense involved in making one order
Y = expected yearly sales
I = inventory carrying cost expressed as percentage.

$$\frac{d(TVC)}{dQ} = \frac{CI}{2} - \frac{YS}{Q^2}.$$

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†The author wishes to thank his colleagues of the Advanced Supply System, Research and Development Division, and in particular is grateful for the encouraging supervision of Mr. J. R. Simpson, Jr.

Setting this equal to zero and solving the resultant equation for Q gives

$$Q = \sqrt{\frac{2YS}{CI}} \tag{2}$$

for the purchase quantity that minimizes cost.

We can find the minimum total variable cost by substituting (2) in (1). This yields

$$(\min)\ TVC = \sqrt{\frac{2YS}{CI}}\ \frac{CI}{2} + YS\sqrt{\frac{CI}{2YS}} = \sqrt{2YSCI}\ . \tag{3}$$

If errors are made in several of the parameters, Y, C, S, I, so that the figures used in computing the optimum purchase quantity are $k_y Y$, $k_c C$, $k_s S$ and $k_i I$, the quantity ordered each time will be

$$Q' = \sqrt{\frac{k_y k_s}{k_c k_i}}\sqrt{\frac{2YS}{CI}}\ . \tag{4}$$

Let

$$k = \frac{k_y k_s}{k_c k_i}\ .$$

The number of times the order will be placed, however, depends on actual demand and not on the demand incorrectly anticipated. (We now assume instantaneous replenishment of stocks upon their reaching zero level.) Orders will therefore be placed $\frac{Y}{Q'}$ times per year.

Costs of order and inventory when the incorrect data is used will be

$$TVC' = \sqrt{\frac{2k\ YS}{CI}}\ \frac{CI}{2} + \sqrt{\frac{CI}{2k\ YS}}\ YS$$

$$= k\sqrt{\frac{YSCI}{2k}} + \sqrt{\frac{YSCI}{2k}} \tag{5}$$

$$= \frac{k+1}{\sqrt{2k}}\ \sqrt{YSCI}\ .$$

The ratio of total variable costs, when one attempts to minimize cost with incorrect data to these costs with correct data, is

$$\frac{k+1}{2\sqrt{k}}\ . \tag{6}$$

Figure 1 graphs the function $g = \frac{k+1}{2\sqrt{k}}$, which is the ratio of total variable costs when one attempts to minimize cost with incorrect data to those costs with correct data. The two rows of figures along the abscissa are reciprocals, and

$$g(k) = \frac{k+1}{2\sqrt{k}} = \frac{1}{2}\left[\sqrt{k} + \frac{1}{\sqrt{k}}\right] = g\left(\frac{1}{k}\right).$$

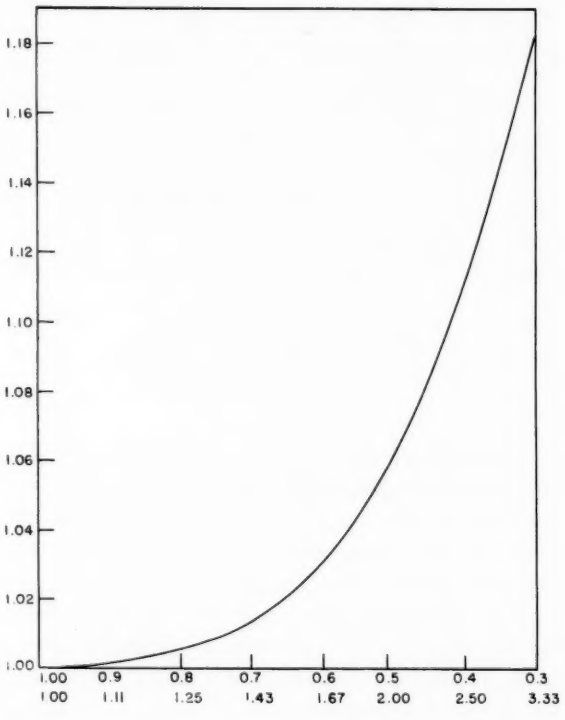

Figure 1

II. CASE OF UNCERTAIN DEMAND

In the following, an algebraic expression will be derived for the proportionate increase in total variable cost resulting from a given proportionate error in the demand data. Demand is assumed to be a random variable with the rectangular distribution the origin of which is at zero. Both order costs and the penalty for depletion are assumed to be linear functions.

Let k = the price of the commodity
p = penalty per unit shortage
a = discount rate per period
$\phi(s)$ = frequency function of demand.

When order costs and penalty for depletion are both linear, the cost of following an optimal inventory policy is:

$$f(x) = \underset{y \geq x}{\text{Min}} \left[k(y-x) + a \left\{ \int_y^\infty p(s-y)\,\phi(s)ds + f(o) \int_y^\infty \phi(s)ds + \int_o^y f(y-s)\,\phi(s)ds \right\} \right].$$

(See Ref. [1], Chapter V, Section 5, pp. 159-164.)

The policy itself is, at each order time, to replenish stock to the level $\bar{x}$, where $\bar{x}$ is the solution to the equation:

$$k - ap \int_{\bar{x}}^\infty \phi(s)ds - ak \int_0^{\bar{x}} \phi(s)ds = 0 .$$

$$k = ap \int_{\bar{x}}^\infty \phi(s)ds + ak \left(1 - \int_{\bar{x}}^\infty \phi(s)ds\right),$$

$$k\,(1-a) = a(p-k) \int_{\bar{x}}^\infty \phi(s)ds ,$$

$$\int_{\bar{x}}^\infty \phi(s)ds = \frac{k(1-a)}{a(p-k)},$$

$$\int_0^{\bar{x}} \phi(s)ds = 1 - \frac{k(1-a)}{a(p-k)} = \frac{ap-k}{a(p-k)} .$$

If $\phi(s)$ is the rectangular distribution with the minimum of the range at zero, it is no real limitation to consider the domain in which it is positive, as from zero to one. Then $\bar{x} = \dfrac{ap-k}{a(p-k)}$.

$$f(o) = k\bar{x} + a \left\{ \int_{\bar{x}}^1 p(s-\bar{x})ds + f(o)\,(1-\bar{x}) + \int_0^{\bar{x}} f(\bar{x}-s)ds \right\}.$$

Since $f(x) = f(o) - kx$, we can rewrite the above as:

$$f(o) = k\bar{x} + a \left\{ \int_{\bar{x}}^1 p(s-\bar{x})ds + f(o)\,(1-\bar{x}) + f(o)\,\bar{x} - k \int_0^{\bar{x}} (\bar{x}-s)ds \right\}$$

or

$$f(o)\,(1-a) = k\bar{x} + a\left\{\int_{\bar{x}}^{1} p(s-\bar{x})ds - k\int_{0}^{\bar{x}} (\bar{x}-s)ds\right\}.$$

Now

$$\int_{\bar{x}}^{1} (s-\bar{x})ds = \left[\frac{s^2}{2} - \bar{x}s\right]_{\bar{x}}^{1} = \frac{1}{2} - \bar{x} - \frac{\bar{x}^2}{2} + \bar{x}^2 = \frac{1}{2}(1-\bar{x})^2,$$

and

$$\int_{0}^{\bar{x}} (\bar{x}-s)ds = \left[\bar{x}s - \frac{s^2}{2}\right]_{0}^{\bar{x}} = \bar{x}^2 - \frac{\bar{x}^2}{2} = \frac{\bar{x}^2}{2}.$$

Therefore,

$$f(o)\,(1-a) = k\bar{x} + a\left(\frac{p}{2}(1-\bar{x})^2 - \frac{k}{2}\bar{x}^2\right).$$

As pointed out above

$$\bar{x} = \frac{ap-k}{a(p-k)} = \frac{a\,p/k-1}{a\,p/k-a}.$$

If p and k are both multiplied by a factor N, then $\bar{x}$ is unchanged and so is the proportion of maximum anticipated demand which will be ordered to. Parenthetically, we remark that the fact that $\bar{x}$ is not changed as a result of multiplying p and k by the same factor N is independent of the form of the distribution function describing demand.

If the upper limit of the range over which demand can extend is incorrectly estimated, this is equivalent to multiplying p and k in the computation of $\bar{x}$ by N, where N is the proportion that the incorrectly estimated range is to the actual range over which demand extends. The number of units (expressed in terms of the units of true maximum demand) that will be ordered to on the basis of the incorrect information will be $\bar{x}' = N\bar{x}$. The cost of following this policy is given by the equation:

$$f(o)_N\,(1-a) = kN\bar{x} + a\left(\frac{p}{2}(1 - N\bar{x})^2 - \frac{k}{2}N^2\bar{x}^2\right)$$

provided $N\bar{x} < 1$.

$$\frac{f(o)_N}{f(o)} = \frac{kN\bar{x} + a\left(\frac{p}{2}(1 - N\bar{x})^2 - \frac{k}{2}N^2\bar{x}^2\right)}{k\bar{x} + a\left(\frac{p}{2}(1-\bar{x})^2 - \frac{k}{2}\bar{x}^2\right)}.$$

Dividing the numerator and denominator of the above by k we obtain:

$$\frac{f(o)_N}{f(o)} = \frac{N\bar{x} + a\left\{\frac{p}{2k}(1 - N\bar{x})^2 - \frac{1}{2}N^2\bar{x}^2\right\}}{\bar{x} + a\left\{\frac{p}{2k}(1-\bar{x})^2 - \frac{1}{2}\bar{x}^2\right\}}.$$

As pointed out above, the value of $\bar{x}$ depends only on the ratio p/k. We can therefore compute $\frac{f(o)_N}{f(o)}$ without knowing the true maximum demand.

We shall now put the expression for the relationship between $f(o)_N$ and $f(o)$ in a form more appropriate to bring out its dependence on p/k and N separately.

$$f(o)_N\,(1-a) = kN\bar{x} + a\left\{\frac{p}{2}(1 - 2N\bar{x} + N^2\bar{x}^2) - \frac{k}{2}N^2\bar{x}^2\right\},$$

$$(1-a)\left[f(o)_N - f(o)\right] = (N-1)\,k\bar{x} + a\left\{\frac{p}{2}\left[-(N-1)\,2\bar{x} + (N^2-1)\,\bar{x}^2\right] - (N^2-1)\,\frac{k}{2}\bar{x}^2\right\},$$

$$= (N-1)\left\{k\bar{x} + a\left[\frac{p}{2}(-2\bar{x} + (N+1)\,\bar{x}^2) - (N+1)\,\frac{k}{2}\bar{x}^2\right]\right\},$$

$$= (N-1)\left\{(k - ap)\,\bar{x} + (N+1)\,\frac{a}{2}(p-k)\,\bar{x}^2\right\}.$$

Since $\bar{x} = \frac{ap-k}{a(p-k)}$, this gives

$$= (N-1)\left\{(k - ap)\,\bar{x} + (N+1)\,\frac{(ap-k)}{2}\,\bar{x}\right\},$$

$$= (N-1)\left\{(N-1)\,\frac{(ap-k)}{2}\,\bar{x}\right\},$$

$$= \frac{(N-1)^2}{2}\,(ap-k)\,\bar{x}.$$

$$\frac{[f(o)_N - f(o)]}{f(o)} = \frac{\frac{(N-1)^2}{2}\,(ap-k)\,\bar{x}}{k\bar{x} + a\left\{\frac{p}{2}(1-\bar{x})^2 - \frac{k}{2}\bar{x}^2\right\}}$$

$$= \frac{\frac{(N-1)^2}{2}\,(a\,p/k-1)\,\bar{x}}{\bar{x} + a\left\{\frac{p}{2k}(1-\bar{x})^2 - \frac{1}{2}\bar{x}^2\right\}}.$$

Letting $\frac{p}{k} = z$ and substituting $\bar{x} = \frac{az - 1}{a(z - 1)}$ in the above, we get:

$$\frac{\frac{(N-1)^2}{2}\,\frac{(az-1)^2}{a(z-1)}}{\frac{az-1}{a(z-1)} + a\left\{\frac{z}{2}\left(\frac{1-a}{a(z-1)}\right)^2 - \frac{1}{2}\left(\frac{az-1}{a(z-1)}\right)^2\right\}} = \frac{(N-1)^2\,(az-1)^2}{2(az-1) + a\left\{z\,\frac{(1-a)^2}{a(z-1)} - \frac{(az-1)^2}{a(z-1)}\right\}},$$

$$= \frac{(N-1)^2\,(az-1)^2\,(z-1)}{2(az-1)(z-1) + \{z(1-a)^2 - (az-1)^2\}},$$

$$= \frac{(N-1)^2\,(az-1)^2\,(z-1)}{(az-1)\{2(z-1) - (az-1)\} + z(1-a)^2},$$

$$= \frac{(N-1)^2\,(az-1)^2\,(z-1)}{(az-1)\,(z(2-a)-1) + z(1-a)^2}.$$

This gives the proportionate increase in cost resulting from a given percentage error in estimating maximum demand as the product of (1) a function of the proportionate error in estimating maximum demand and (2) a function of the ratio of penalty for depletion per unit to unit cost of acquisition and the discount factor.

In Part I, demand was assumed to be a fixed quantity, not a random variable. The proportionate increase in cost was obtained as a function of the proportionate error in the data and found to be independent of the true value of any of the parameters needed to formulate the policy. When demand is a stochastic variable, we are not able to obtain such a convenient relation.

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1. Bellman, R. I., Dynamic Programming, Princeton University Press (1957).
2. Whitin, Thomson M., The Theory of Inventory Management, Princeton University Press (1953).

* * *

SOME QUEUING PROBLEMS IN MACHINE MAINTENANCE*

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This paper discusses some queuing problems that arise in the context of the maintenance of a group of machines, each of which is operated intermittently and all of which are subject to several kinds of malfunctions. A general formulation of this problem is made and the general method of solution indicated. A class of solutions in form suitable for direct computations is given. As a special case, a solution is given to the case of m_j repairmen of type j, j = 1, ..., k for a group of fully utilized machines.

THE GENERAL CASE

System States

Let $i_j \equiv$ the number of machines requiring repair of type j, j = 0, 1, ..., k, where a repair of type 0 means that no repair is needed. Then $(i_0, i_1, \ldots, i_k)$ represents the state of serviceability of a group of N machines, under the constraint

$$\sum_{j=0}^{k} i_j = N . \tag{1}$$

The totality of possible states of the system can be thought of as the set of points with integer coordinates contained in the hyper-tetrahedron whose vertices are the points: (N, 0, ..., 0), (0, N, ..., 0), ..., (0, ..., 0, N). Impossible states of the system correspond to points lying outside the hyper-tetrahedron. Impossible changes of state of the system correspond to changes from points contained in the hyper-tetrahedron, to points outside and conversely.

Repair of Malfunctions

Consider a fixed allocation of resources to the repair activity. We can conceive of this as a vector, $(y_1, \ldots, y_h)$, each component of which represents the quantity of some resource, such as skills, skill levels, tools, test equipment, facilities, etc. The way in which such resources are allocated to the accomplishment of unscheduled repairs will be referred to as an assignment policy. For a fixed assignment policy, repairs will be accomplished on the machines out of service because of malfunctions of various types at rates that, in general, depend on the state of the system, as well as on the given set of maintenance resources.

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Let $\delta \mu^{(j)}_{i_0, \ldots, i_k} \equiv$ the probability that a single repair of type j, j = 1, ..., k, is completed in an arbitrary time interval of length δ, when the system is in state $(i_0, \ldots, i_k)$ at the beginning of the interval, and let $o(\delta) \equiv$ the probability that more than one repair is completed in an arbitrary time interval of length δ, where, as usual, $o(\delta)$ indicates a probability of smaller order of magnitude than δ.

Note that a change in the maintenance resource vector or a change in the assignment policy will be manifested in the functions, $\mu^{(j)}_{i_0, \ldots, i_k}$; j = 1, ..., k. It also should be noted that this formulation of the problem excludes from consideration many assignment policies, for example, priority assignments for repairs of individual machines requiring repairs of the same type.

Generation of Malfunctions

The operational requirements for the machines might be such that although it is desirable to have numbers of machines on hand requiring no repairs, not all of the machines requiring no repairs necessarily need be in operation at an arbitrary time. Such a situation occurs in many military applications. Therefore the rate at which malfunctions occur will be written as a function of the state of the system and not necessarily as proportional to the number of machines requiring no repairs.

Let $\delta \lambda^{(j)}_{i_0, \ldots, i_k} \equiv$ the probability that a single malfunction of type j, j = 1, ..., k occurs in an arbitrary time interval of length δ when the system is in state $(i_0, \ldots, i_k)$ at the beginning of the interval, and let $o(\delta) \equiv$ the probability that more than one malfunction of any type occurs in an arbitrary time interval of length δ.

Note that a change in the utilization policy of the group of machines will be manifested in the functions $\lambda^{(j)}_{i_0, \ldots, i_k}$; j = 1, ..., k. It also should be noted that this formulation excludes many utilization policies from consideration, for example, differential utilization of individual machines.

General Equations

By the usual methods [1] it can be shown that the ergodic probabilities, $p_{i_0, \ldots, i_k}$, satisfy the following set of $\frac{(N + k)!}{N!\, k!} + 1$ set of linear equations:

$$\sum_{j=1}^{k} \left\{ \lambda^{(j)}_{i_0, \ldots, i_j - 1, \ldots, i_k} \, p_{i_0, \ldots, i_j - 1, \ldots, i_k} + \mu^{(j)}_{i_0, \ldots, i_j + 1, \ldots, i_k} \, p_{i_0, \ldots, i_j + 1, \ldots, i_k} \right\} - p_{i_0, \ldots, i_k} \sum_{j=1}^{k} \left(\lambda^{(j)}_{i_0, \ldots, i_j, \ldots, i_k} + \mu^{(j)}_{i_0, \ldots, i_j, \ldots, i_k} \right) = 0 . \tag{5}$$

$$(2) \qquad \sum_{i_0, \ldots, i_k = 0}^{N} p_{i_0, \ldots, i_k} = 1 ,$$

where $i_0, \ldots, i_k = 0, \ldots, N$, and $\sum_{j=0}^{k} i_j = N$, and where $p_{i_0, \ldots, i_k} = 0$ for impossible states of the system, and where $\lambda^{(j)}_{i_0, \ldots, i_k}$ and $\mu^{(j)}_{i_0, \ldots, i_k} = 0$ for respectively impossible changes in state of the system.

The general solution quickly involves vast quantities of computations as the number of machines and number of types of malfunctions increase. (For 20 machines and 3 types of repairs, there are 1772 equations in the same number of unknowns.) For some special cases of interest, however, solutions can be found which are suitable for direct computations.

A CLASS OF SOLUTIONS

Assumptions

For many applications of interest, the machine utilization policy is dependent only on (but not necessarily proportional to) i_0, the number of machines requiring no repairs. When this is so, the rate of occurrence of malfunctions can be written

$$(3) \qquad \lambda^{(j)}_{i_0, \ldots, i_k} = \lambda^{(j)}_{i_0} \; ; \; j = 1, \ldots, k .$$

Further, it is natural to assume that the malfunction rates of various types will be pairwise proportional for all i_0, therefore,

$$(4) \qquad \lambda^{(j)}_{i_0} = \left\{ \begin{array}{ll} \theta_j \lambda_{i_0} & ; \; i_0 \pm 0 ; \\ 0 & ; \; i_0 \neq 0 ; \end{array} \right\} \theta_j > 0 \; ; \; j = 1, \ldots, k .$$

Also, the maintenance resources vector, the assignment policy, or both may be such that the rate at which repairs of type j are accomplished depend only on i_j, the number of machines requiring such repairs. (This would hold, e.g., for organizations which adhere to so called specialized maintenance policies.) For this situation the repair rates become

$$(5) \qquad \mu^{(j)}_{i_0, \ldots, i_k} = \left\{ \begin{array}{ll} \mu^{(j)}_{i_j} & i_j \pm 0 ; \\ 0 & i_j \neq 0 ; \end{array} \right\} \quad j = 1, \ldots, k .$$

Solution

In this case, the general equations (2) have the solution:

$$p_{i_0, \ldots, i_k} = \frac{\prod_{j=1}^{k} \theta_j^{i_j} \prod_{h=i_0+1}^{N} \lambda_h}{\prod_{j=1}^{k} \prod_{h=1}^{i_j} \mu_h^{(j)}} p_{N, 0, \ldots, 0} \; ; \; i_0 < N , \tag{6}$$

where

$$p_{N, 0, \ldots, 0}^{-1} = \sum_{i_0, \ldots, i_k = 0}^{N} \left(\frac{p_{i_0, \ldots, i_k}}{p_{N, 0, \ldots, 0}} \right)$$

where the summation is restricted by

$$\sum_{j=0}^{k} i_j = N .$$

Various Machine-Utilization Policies

Proportional utilization. For many applications, it is useful to assume that the machine-utilization policy is such that the rate of malfunction is proportional to the number of serviceable machines. (This would be the case for complete utilization, for example.) In this case, λ_h may be written as

$$\lambda_h = h\lambda \; ; \; h = 1, \ldots, N , \tag{7}$$

and equation (6) becomes

$$p_{i_0, \ldots, i_k} = \frac{\frac{N!}{i_0!} \lambda^{N-i_0} \prod_{j=1}^{k} \theta_j^{i_j}}{\prod_{j=1}^{k} \prod_{h=1}^{i_j} \mu_h^{(j)}} p_{N, 0, \ldots, 0} \; ; \; i_0 < N . \tag{8}$$

Fixed nominal utilization. In some applications, it is desired to fix the utilization rate at some constant nominal level which is independent of the number of machines (except when the number of serviceable machines is insufficient to achieve the fixed level). In this case, λ_h may be written as

$$\lambda_h = \begin{cases} \lambda^* & ; \; h \geq h^* \\ \lambda_h < \lambda^* & ; \; h = 1, \ldots, h^* - 1 \end{cases} \tag{9}$$

and equation (6) modified accordingly. When the nominal utilization can be achieved by one serviceable machine, $h^* = 1$ and equation (6) becomes

$$(10) \qquad p_{i_0, \ldots, i_k} = \frac{\prod_{j=1}^{k} \theta_j^{i_j} \, (\lambda^*)^{N-i_0}}{\prod_{j=1}^{k} \prod_{h=1}^{i_j} \mu_h^{(j)}} \, p_{N, 0, \ldots, 0} \; ; \; i_0 < N \, .$$

This solution is of value in particular applications wherein a high serviceability rate is considerably more important than a high utilization rate, as in the case of fire fighting apparatus, for example.

Various Maintenance-Scheduling Policies

The repair team. If the resources allocated to repair of malfunctions and the repair-assignment policy are such that, for each type of malfunction, there are a fixed number of identical "bundles" of men, tools, etc., to be assigned to repair activities, the expression for $\mu_{i_j}^{(j)}$ becomes

$$(11) \qquad \mu_{i_j}^{(j)} = \left\{ \begin{array}{ll} i_j \, \mu^{(j)} & ; \; i_j \leq m_j \; ; \\ m_j \, \mu^{(j)} & ; \; i_j > m_j \; ; \end{array} \right\} \; j = 1, \ldots, k \, .$$

This is the "dock" concept of maintenance, with m_j "docks" ("teams", etc.) of type j. When the machines are fully utilized, equations (8) and (11) give the solution to this case, which is also the case of m_j repairmen of type j; j = 1, ..., k, with different servicing times and fully utilized machines.

One repair team of each type. When $m_j \equiv 1$ in the above discussion, the solution [2] of the case of fully utilized machines with k repairmen of different types, each with different servicing times, is obtained.

REFERENCES

[1] Feller, W., An Introduction to Probability Theory and Its Applications, Wiley (1950), Ch. 17.

[2] Benson, F., and Cox, D. R., "The Productivity of Machines Requiring Attention at Random Intervals," Journal of the Royal Statistical Society, Series B, Vol. 13, pp. 65-82, 1951.

* * *

NOTES

PROBLEMS

It has been suggested that this journal might serve as a medium for publishing problems in the area of Logistics – with the idea that other persons might be interested in those problems and might submit comments. Readers are invited to submit brief statements on applied and theoretical problems in Logistics. Address letters to Managing Editor, Naval Research Logistics Quarterly, Office of Naval Research, Washington 25, D. C.

NEWS AND MEMORANDA

Readers are invited to submit to the Managing Editor items of general interest in the field of logistics

Dr. F. D. Rigby has assumed the position of Director, Mathematical Sciences Division, Office of Naval Research. He relieves Dr. F. J. Weyl who has been appointed to the position of Director, Naval Analysis Group, Office of Naval Research. Dr. Rigby also continues as Head, Logistics Branch on an interim basis.

RECENT PUBLICATIONS

EDITOR'S NOTE: Listing of a publication in this section is for record and reference only and does not constitute an endorsement of point of view or advocacy of use.

PRINCIPLES AND TECHNIQUES OF APPLIED MATHEMATICS. By Bernard Friedman, John Wiley and Sons, Inc., New York, 1956.

A major motivation in the progress of mathematics is a mathematician's fear of unwittingly repeating the same reasoning in different contexts. Bernard Friedman, in Principles and Techniques of Applied Mathematics, furnishes a nice example in the exposition of such progress with an introduction to the theory of linear operators and its conceptual unification of a wide variety of problems in differential and difference equations, linear systems, etc.

Written, and used, as a graduate-level text, it develops methods for solving problems in linear operators in ways so that the analogy to systems of linear equations is easy to keep in mind. This feature seems worthwhile and well done. Another "plus" for the student is given in the way Professor Friedman introduces new concepts, quite consistently supplying an initial heruistic, motivating anchor to the reader's past experience in advance of formal mathematical definitions.

The one shortcoming in an otherwise excellent book, in the reviewer's opinion, is in the handling of the Dirac "δ function" and related topics, such as L. Schwartz's theory of distributions. In the author's defense, it must be pointed out that prerequisities of the book do not include a knowledge of Lebesgue integration, and measure theory, so some compromise is necessary. However, as a particular criticism, "functions" are given without defining a mapping for every point of the domain – e.g.,

$$\text{"}H(x) = 1, x > 0$$
$$= 0, x < 0\text{"}$$

And, in general, concepts such as ($\delta(x)$ is the Dirac "function" here): "... $\delta(x)$ is treated as if it were an ordinary function except that we shall never talk about 'values' of $\delta(x)$ ... " seems to lack the standards of clarity and crispness which the author has set in most of the book.

Overall, Principles and Techniques in Applied Mathematics brings a unified account of a powerful section of modern mathematics within reach of graduate-level reading and is a welcome addition to the literature.

Harlan Mills

ECONOMIC MODELS: AN EXPOSITION. By E. F. Beach, John Wiley and Sons, Inc., New York, 1957.

In this book Professor Beach offers a short course in two parts on some of the mathematical and statistical techniques which are the basis for economic model building. Part I covers the mathematics of abstract models. The topics covered include: the nature of equations, linear models, nonlinear models, continuous dynamic models (involving differential equations), and sequence models (involving difference equations and their solution). Part II concerns itself with the mathematics of models based on actual statistical data. The discussion includes: the elements of simple statistical theory, simple regression theory, multivariate regression, and the theory of multiple relations. Sample problems and bibliography are provided for each chapter.

W. G. Mellon

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